Formulation of a Basic Building Block Model for Interaction of High Speed Vehicles on Flexible Structures

L. Vu-Quoc
Department of Aerospace Engineering, Mechanics, and Engineering Science, University of Florida, Gainesville, Fla. 32611
Member ASME

M. Olsson
Division of Structural Mechanics, Lund University, S-221 00 Lund, Sweden

1 Introduction

In recent years considerable interest has been developed in implementing energy-efficient, high-speed, low-noise systems for airport-city or intercity transportation—in particular, the magnetically levitated (Maglev) vehicle systems (cf. Eastham and Hayes (1987)). Currently, to ensure success of Maglev systems, guideway structures must be designed to be stiff so that deflections remain within narrow margins of tolerance. The cost of a stiff guideway structure can easily exceed 70 percent of the total cost of a system (Zicha (1986)). More flexible guideways are less expensive, but present complex vehicle/structure interaction.\(^1\) The interaction between high speed moving vehicles and flexible supporting structures is the focus of the present paper. Even though the impetus behind this work is geared toward high speed vehicles, the problem of moving loads does find applications in various fields of engineering (cf. Fryba (1972), Blejwas et al. (1979)). Extensive lists of references on the subject of moving loads over elastic structures are contained in the classical monograph by Fryba (1972), and in several review papers, e.g., Kortüm and Wormley (1981), Ting and Yener (1985), report of Subcommittee on Vibration Problems (1985) and Kortüm (1986).

\(^1\)Progress in suspension control technology will make possible the use of flexible guideways, and the efficiency of Maglev systems will increase with advance in superconductor research.


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We note that the wheel model also finds application in electrodynamic (repulsive) Maglev vehicles since these vehicles move on wheels up to a maximum lift-off speed of about 80 km/h (Alscheter et al. (1983)). Further, both high speed Maglev vehicles and wheel-on-rail vehicles may possibly run on the same bivalent guideway structure.

Nonlinear equations of motion of the basic model, valid for large deformation of the beam, are derived for a class of general (nonlinear) contact constraints via Hamilton’s principle of stationary action. In the present work, structural response in the small deformation range is, however, our main interest. With assumptions of small deformation, the nonlinear equations of motion are then reduced, in a consistent manner, to a system of mildly nonlinear equations. This consistency is an important feature that distinguishes the present approach from traditional practice of complete linearization: Nonlinear terms of physical relevance, essential for high speed regime, are retained in the equation for nominal motion of the basic model. Finally, an example of vehicle/structure interaction at different initial velocities is given to illustrate the present formulation.

Note that the study of dynamic motion of the complete system, driven by external forces, as done here, is the only way to explain the Timoshenko paradox: Consider a constant vertical force traversing, with some prescribed motion, a simply-supported beam. Since the net work done by the force is zero, where does the energy which leaves the beam in a vibratory state after the traversing come from? The same question can be asked for a moving mass with prescribed motion. In fact, the “excised” of energy comes precisely from the work done by (unmodeled) external forces needed for the vehicle to follow the motion prescribed (cf. Maunder (1960)).

2 Description of Basic Problem

In this section, we describe the basic problem of planar motion of a high speed moving load—a single rigid wheel or a suspended magnet with tight gap control—over a flexible beam. Attention is focused, however, to the dynamics of the more complex case of a rolling wheel. Several possible models of a Maglev magnet (“magnetic wheel”) can be obtained from this basic model. Recall that the present basic model serves as a building-block for more complex vehicle/structure models.

2.1 Basic Assumptions. Let \( \{E_1, E_2\} \) be orthonormal basis vectors, and \( \{X^1, X^2\} \) the coordinates along these axes. These objects define a coordinate system for the material (undeformed) configuration of a beam. The line of centroids of the beam, of length \( L \) and initially straight, is assumed to lie along the axis \( E_1 \); the coordinate of a material point on the line of centroids is denoted by \( S = X^1 \in [0, L] \). Let \( \{e_i, e_j\} \) be the set of orthonormal vectors spanning the spatial (deformed) configuration, and conveniently chosen so that \( E_i = e_i \), for \( i = 1, 2 \). The displacement of a material point \( S \) is denoted by \( u(S, t) = u_e(S) e_e, \) where \( t \in [0, +\infty) \) is the time parameter.

Consider a rigid wheel with mass \( M \), radius \( R \), and rotatory inertia about its center of mass \( I_m \). Let \( Y(t) = Y^a(t) E_a \) be the position of the wheel center of mass in the material configuration of the beam; its position in the spatial configuration is denoted by \( y(t) = y^a(t) e_a \). We consider the following general form of constraint

\[
y^a(t) = y^a(t) + g^a(u(Y^1(t), t), u_5, Y^1(t), t), \tag{1}
\]

for \( \alpha = 1, 2 \), where \( g^a (\cdot, \cdot) \) are some functions of the structural displacement \( u \) and its spatial derivative \( u_5 = \frac{\partial u}{\partial S} = \left( \frac{\partial u^a}{\partial S} \right) e_a \), such that \( g^a(0, 0) = 0 \). We call \( Y(t) \), the motion of the wheel in the material configuration of the beam, the nominal motion of the wheel. Thus, for \( u(S, t) = 0 \), we have \( y(t) = Y(t) \). Given the functions \( y(t), u(S, t), \) and \( g^a(u, u_5) \), relation \( (1) \) with \( \alpha = 1 \) could be taken as a definition of the (unknown) nominal motion \( Y(t) \), i.e., \( Y(t) \) is defined to be a solution of \( (1) \). In this formulation, we consider only the case where \( y^2 = R \), for some constant \( R \). Let \( \theta \) denote the angle of revolution of the wheel, which is considered to be a function of the nominal position \( Y(t) \) and the structural deformation \( \{u, u_5\} \).

We will often employ the shorthand notation \( g^a(Y^1, t) = g^a(u(Y^1, t), u_5(Y^1, t)), \) and similarly with \( \theta(Y^1, t) = \theta(Y^1, u_5(Y^1, t)) \). Thus,

\[
\frac{\partial \theta}{\partial S} = \frac{\partial \theta}{\partial S} + \frac{\partial \theta}{\partial u} \frac{\partial u}{\partial S} + \frac{\partial \theta}{\partial u_5} \frac{\partial u_5}{\partial S}, \tag{2a}
\]

\[
\frac{\partial \theta}{\partial S} = \frac{\partial \theta}{\partial S} + \frac{\partial \theta}{\partial u} \frac{\partial u}{\partial S} + \frac{\partial \theta}{\partial u_5} \frac{\partial u_5}{\partial S}. \tag{2b}
\]

2.2 Kinetic Energy and Potential Energy. The kinetic energy \( K \) of the basic system (wheel and flexible beam) is given by

\[
K = \frac{1}{2} \int_{[0, L]_S} \left\{ \frac{\partial Y^1}{\partial t} \cdot \frac{\partial Y^1}{\partial t} + \frac{\partial Y^2}{\partial t} \cdot \frac{\partial Y^2}{\partial t} \right\} \frac{1}{2} I_w \left( \frac{\partial \theta}{\partial t} \right)^2 + \frac{1}{2} \int_{[0, L]_S} A_S \left[ \left( \frac{\partial u}{\partial S} \right)^2 + \frac{\partial u_5}{\partial S} \right] \frac{1}{2} I_w \left( \frac{\partial \theta}{\partial t} \right)^2 \tag{3}
\]

where the superposed \( * \) denotes the total time derivative (i.e., \( \frac{d}{dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \frac{\partial}{\partial S} \frac{dS}{dt} \); \( \frac{\partial u}{\partial S} = \frac{\partial u}{\partial S} \) denotes the partial derivative of \( u \) in time, and \( A_S \) the mass per unit length of the beam. Now, consider a function \( f: [0, L] \times [0, \infty) \rightarrow \mathbb{R} \), smooth enough in both arguments. The first and second total time derivatives of \( f(S, t) \), evaluated at \( S = Y^1(t) \), are obtained as follows

\[
\frac{df(Y^1, \theta^1, t)}{dt} = \frac{\partial f(Y^1, \theta^1, t)}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial f(Y^1, \theta^1, t)}{\partial \theta^1} \frac{d\theta^1}{dt} + \frac{\partial f(Y^1, \theta^1, t)}{\partial Y^1} \frac{dY^1}{dt} \tag{4a}
\]

\[
\frac{df(Y^1, \theta^1, t)}{dt} = \frac{\partial f(Y^1, \theta^1, t)}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial f(Y^1, \theta^1, t)}{\partial \theta^1} \frac{d\theta^1}{dt} + \frac{\partial f(Y^1, \theta^1, t)}{\partial Y^1} \frac{dY^1}{dt} + \frac{\partial^2 f(Y^1, \theta^1, t)}{\partial \theta^2} \left( \frac{d\theta}{dt} + \frac{d\theta^1}{dt} \right)^2 \tag{4b}
\]

We will often omit to specify \( (\theta, \theta^1) \) in the argument lists of quantities such as \( f \) and \( f \), and simply write \( f(Y^1, \theta) \) and \( f(Y^1, \theta^1) \). Thus, employing \( (4a) \) and \( (4b) \) with \( f = g^a \) to evaluate \( g^a(Y^1, \theta) \) and \( \theta(Y^1, t) \), one obtains an expanded form of the kinetic energy \( K \). The convexive terms in \( (4) \)—i.e., the first term in \( (4a) \), and the first three terms in \( (4b) \)—play an important role in the interaction between high speed moving vehicles and the supporting flexible structures, as shown in Bleijwas, Feng, and Ayre (1979), where numerical results corroborated experimental findings (see also Ting, Genin, and Ginsberg (1974)). Further, by the assumed smoothness of the function \( f \) in \( (4) \), total time derivatives and spatial derivatives are interchangeable,

\[
\frac{d^2 f(Y^1, \theta, \theta^1, t)}{dS^2} = \frac{d^2 f(Y^1, \theta, \theta^1, t)}{dS^2} \tag{5}
\]

\[1\] The term “contact” is also used here for Maglev magnets with tight gap control.

\[2\] Throughout the paper, summation convention is implied on repeated indices, which take values in \( [1, 2] \).

\[3\] In this paper, the notation used is \( x^a(t) = x(t) \).

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and thus notation such as \( f_y(Y^1),t \) can be used without confusion.

The wheel is subjected to an applied force \( \mathbf{F} = F^e \mathbf{e}_a \), and a torque \( T \) about its center of mass. Without loss of generality, for the moment, the applied force and torque can be considered constant in time for the purpose of deriving the equations of motion. The work done by the external forces is then given by \( W := \mathbf{F} \cdot \mathbf{r} + I T \). Further, let \( \psi(u) \) denote the elastic strain energy stored in the beam. The formulation is so far valid for large deformation in the beam, and we have not yet introduced assumptions of small deformation at this stage. Explicit expression of \( \psi(u) \) for finite deformation of a beam in plane motion can be found in Simo and Vu-Quoc (1986).

### 3 Derivation of Equations of Motion

In this section, we derive the equations of motion for the basic problem, valid for large structural deformation, by employing Hamilton’s principle of stationary action. Additional assumptions of small deformation in the structure are subsequently introduced to further simplify the equations of motion. This simplification process is carefully carried out in a manner that is consistent with the assumptions. It should be indicated that even though particularized to small structural deformation the resulting equations of motion do retain some crucial nonlinear terms, for an adequate description of the dynamics at high speed regime.

3.1 The General Nonlinear Equations of Motion.

The Lagrangian of the system can be written as

\[
\mathcal{L}(Y,\psi) = K(Y,\psi) - \psi(u) + W(Y,\psi),
\]

(6)

Consider the time interval \( [t_1,t_2] \). Let \( (\psi(t),\eta(S,t),\eta(S,t)) \) be the admissible variations corresponding to the functions \( Y^1, u^1, u^2 \), and vanishing at time \( t = t_1 \) and \( t = t_2 \). The equations of motion are derived from the stationary condition of the action integral, i.e., the Euler-Lagrange equations corresponding to (6):

\[
\frac{d}{de} \int_{[t_1,t_2]} \mathcal{L}(Y^1 + \epsilon \psi, u + \epsilon \eta) dt \bigg|_{\epsilon = 0} = 0,
\]

(7)

for all admissible variations \( (\psi, \eta) \), where \( \eta = \eta^2 \mathbf{e}_g \). It follows that the equations for nominal motion \( Y^1 \) and for structural displacement \( u \) are, respectively, given by

\[
\frac{d}{de} \int_{[t_1,t_2]} \mathcal{L}(Y^1 + \epsilon \psi, u + \epsilon \eta) dt \bigg|_{\epsilon = 0} = 0,
\]

(8)

### Nominal Motion \( Y^1 \).

We first note that from (4a) one has

\[
\frac{\partial f}{\partial \psi}(Y^1,\dot{Y}^1,t) = \frac{\partial f}{\partial \psi}(Y^1,t).
\]

(9a)

Then, it follows from (9a) and (5) that

\[
\frac{d}{dt} \left( \frac{\partial f(Y^1,t)}{\partial \psi} \right) - \frac{d}{dt} \left( \frac{\partial f(Y^1,t)}{\partial S} \psi \right) \frac{\partial f(Y^1,t)}{\partial S} \psi,
\]

(9b)

Further, the variation of \( f \) with respect to \( Y^1 \) is given by

\[
\frac{d}{de} f(Y^1 + \epsilon \psi, u + \epsilon \eta) \bigg|_{\epsilon = 0} = \frac{\partial f(Y^1,t)}{\partial S} \psi,
\]

(10)

where we have made use of (4a).\(^7\) Next, after evaluation of the directional derivative in (8)\(_2\), and applying integration by parts with the boundary conditions \( \psi(t_1) = \psi(t_2) = 0 \), we obtain

\[
-\frac{d}{dt} \int_{[t_1,t_2]} K(Y^1 + \epsilon \psi, u) dt \bigg|_{\epsilon = 0} = \int_{[t_1,t_2]} \left( M \left( 1 + \frac{\partial g(Y^1,t)}{\partial S} \right) \left[ \dot{Y}^1 + \ddot{g}(Y^1,t) \right] + \frac{M \partial g^2(Y^1,t)}{\partial \psi} \ddot{g}(Y^1,t) + \frac{\partial \theta Y^1,t}{\partial g}(Y^1,t) \psi \right) dt,
\]

(11a)

\[
\frac{d}{dt} \int_{[t_1,t_2]} W(Y^1 + \epsilon \psi, u) dt \bigg|_{\epsilon = 0} = \int_{[t_1,t_2]} \left( F^1 \left( 1 + \frac{\partial g(Y^1,t)}{\partial S} \right) + \frac{F^1 \partial g^2(Y^1,t)}{\partial \psi} \ddot{g}(Y^1,t) + \frac{F^1 \partial \theta Y^1,t}{\partial g}(Y^1,t) \psi \right) dt,
\]

(11b)

where use has been made of (9) and (10) with \( f = \psi^2 \) to allow cancellation of certain terms. The stationary condition (8), and relations (11) yield the equation for the nominal motion \( Y^1 \):

\[
M \left( 1 + \frac{\partial g^2(Y^1,t)}{\partial S} \right) \left[ \dot{Y}^1 + \ddot{g}(Y^1,t) \right]
\]

\[
+ M \frac{\partial g^2(Y^1,t)}{\partial S} \ddot{g}(Y^1,t) + \frac{\partial \theta Y^1,t}{\partial g}(Y^1,t) \psi
\]

\[
= F^1 \left( 1 + \frac{\partial g^2(Y^1,t)}{\partial S} \right) + F^1 \frac{\partial g^2(Y^1,t)}{\partial \psi} \ddot{g}(Y^1,t) + \frac{F^1 \partial \theta Y^1,t}{\partial g}(Y^1,t) \psi.
\]

(12)

### Structural Motion \( (u^1,u^2) \).

Similar to relations (9), one can prove that the following identifies hold

\[
\frac{\partial g^2(u^1,u^2)}{\partial \theta^2} \psi = \frac{\partial g^2(u^1,u^2)}{\partial \theta^1} \psi,
\]

(13a)

\[
\frac{d}{dt} \left( \frac{\partial g^2(u^1,u^2)}{\partial \theta^1} \right) = \frac{\partial g^2(u^1,u^2)}{\partial \theta^1} \psi,
\]

(13b)

\[
\frac{\partial g^2(u^1,u^2)}{\partial \theta^2} \psi = \frac{\partial g^2(u^1,u^2)}{\partial \theta^1} \psi,
\]

(13c)

\[
\frac{d}{dt} \left( \frac{\partial g^2(u^1,u^2)}{\partial \theta^2} \psi \right) = \frac{\partial g^2(u^1,u^2)}{\partial \theta^2} \psi.
\]

(13d)

Now, computation of the directional derivative in (8)\(_2\), and integration by parts with respect to time yield the following results

\[
-\frac{d}{dt} \int_{[t_1,t_2]} K(Y^1,u + \epsilon \eta) dt \bigg|_{\epsilon = 0} = \int_{[t_1,t_2]} \left( M \frac{\partial g^2(Y^1,t)}{\partial S} \ddot{g}(Y^1,t) + \frac{\partial \theta Y^1,t}{\partial g}(Y^1,t) \psi \right) dt
\]

\[
+ \eta^2 S \left( Y^1,t \right) \frac{\partial g^2(Y^1,t)}{\partial \theta^2} \psi + \frac{\partial \theta Y^1,t}{\partial g}(Y^1,t) \psi + \eta^2 S \left( Y^1,t \right) \frac{\partial g^2(Y^1,t)}{\partial \theta^2} \psi + \frac{\partial \theta Y^1,t}{\partial g}(Y^1,t) \psi
\]

\[
+ \frac{\partial \theta Y^1,t}{\partial g}(Y^1,t) \psi \right) dt + \int_{[t_1,t_2]} A \eta^2 \mu^2 dS dt,
\]

(14a)

\[
= \int_{[t_1,t_2]} \left( F^1 \left( 1 + \frac{\partial g^2(Y^1,t)}{\partial S} \right) + F^1 \frac{\partial g^2(Y^1,t)}{\partial \psi} \ddot{g}(Y^1,t) + \frac{F^1 \partial \theta Y^1,t}{\partial g}(Y^1,t) \psi \right) dt.
\]

(14b)
where we have made use of the (homogeneous) boundary conditions of (14a, 15a), and the identities (13). Next, let the weak form of the stiffness operator be denoted by $G(u, \eta)$, and

$$G(u, \eta) = \frac{d}{d \xi} \psi(u + c \psi) \bigg|_{\xi = 0}, \quad \xi = 0,$$

where we recall that $\psi(u)$ designates the strain energy of the beam—see Vu-Quoc (1986) and Simo and Vu-Quoc (1986) for an expression of $G(u, \eta)$. Therefore, using (34), (14), and (15a), the weak form of the structural equations of motion is then given by

$$\left[-F + M \dot{Y} + \ddot{g}(Y, t)\right] \psi'(Y, t) \psi'(Y, t) - \psi''(Y, t) \psi''(Y, t) + \left[-F + 2 \dot{M} \ddot{g}(Y, t)\right] \psi'(Y, t) \psi'(Y, t) + \left[-F + M \dddot{g}(Y, t)\right] \psi'(Y, t) \psi'(Y, t) + \left[-T + I \mathbf{J}(Y, t)\right] \left[\eta(Y, t) \frac{\partial Y(t)}{\partial \xi} + \psi^2(Y, t) \frac{\partial Y(t)}{\partial \xi} \right] \psi'(Y, t) \psi'(Y, t) + \int_{[0, L]} A \eta \psi'(S, t) \psi'(S, t) ds + G(u, \eta) = 0, \quad \forall \text{admissible } \eta.$$  \hspace{1cm} (15b)

The corresponding partial differential equations of motion can be easily obtained from (15) by integrating by parts in $S$, and by invoking the fundamental lemma of calculus of variations. We prefer, however, to retain the structural equations of motion in its weak form for numerical work.

**Remark 3.1. Energy Balance.** The balance of system energy at time $t$ can be written as follows

$$h_k + h_l - \int_0^t \left[F(t) \dot{Y}(t) + T(t) \dot{\psi}(t)\right] dt = \mathcal{K}_0 + \psi^2,$$  \hspace{1cm} (16)

where $\mathcal{K}_0$ is as given in (3), $\psi_l$ as given in Simo and Vu-Quoc (1986); the integral term is the work done by (time-varying) external forces. On the right-hand side of (16) are, respectively, the initial kinetic energy $\mathcal{K}_0$ and the initial potential energy $\psi^2$. The discrete form of the system energy balance (16) has proved to be a very useful criterion in the design of reliable numerical integration algorithms for the equations of motion; see Vu-Quoc and Olson (1987, 1988a) for the details.

### 3.2 Contact Constraints and Contact Forces.

The wheel is assumed to be in permanent contact with, and rolling without slipping on, the beam. Clearly, without structural deformation $(u(S, t) = 0)$, the rotation of the wheel is related to its nominal motion by $\dot{\psi} = Y + \dot{R}$. Let $\dot{R}$ denote the distance from the beam centroidal line to the center of mass of the wheel (Fig. 1). For $\dot{R} = R$, the wheel is moving with its circumference tangent to the beam centroidal line. An explicit form of the function $g(t)$ in the general constraint equations (1) for wheel/beam contact, or magnet/magnet with constant gap,

\frac{1}{sinh} \frac{u}{u'} = \frac{\dot{R}}{\dot{R} + R},

where $\dot{Y}(t) = Y(t)$ and $\dot{Y}(t) = Y(t)$, we obtain yet another model (B) of a moving magnet. In practice, even simpler constraints are chosen (model C) so that $Y(t) = Y(t) = Y(t)$ (cf., e.g., Wallrafl (1986)). Thus, there is no direct coupling between vehicle nominal motion and structural axial deformation. In this case, the equations of motion (12) and (15) in weak form simplify to

$$M \dddot{Y} + \frac{\partial^2 \dddot{Y}}{\partial S^2} - F + 2 \dot{M} \dddot{Y} = 0,$$  \hspace{1cm} (18a)

and

$$\eta^2(Y, t)[\ddot{Y} + \frac{\partial^2 \dddot{Y}}{\partial S^2}] + \int_{[0, L]} A \eta \psi'(S, t) \psi'(S, t) ds + G(u, \eta) = 0,$$  \hspace{1cm} (18b)

which are also valid for a finitely deformed beam. In (18), the equation for axial displacement and the equation for transverse displacement are coupled through the nonlinear nature of $G(u, \eta)$ for the finite deformation case. It is important to quantify the (dynamic) contact forces. In particular, these forces are crucial in studying structural response to emergency braking of a vehicle. For the basic problem considered herein, let $F_c = F_c \eta$ be the contact force acting on the wheel. Once $Y(t)$ and $u(t)$ have been solved for, the contact force can be evaluated by $F_c = \mathcal{F} - M \dot{Y}$, obtained from considering the equilibrium of forces acting on the wheel. Recall that $Y(t)$ is

\begin{align}
\frac{1}{sinh} \frac{u}{u'} &= \frac{\dot{R}}{\dot{R} + R},
\end{align}

where $\dot{Y}(t) = Y(t)$ and $\dot{Y}(t) = Y(t)$, we obtain yet another model (B) of a moving magnet. In practice, even simpler constraints are chosen (model C) so that $Y(t) = Y(t) = Y(t)$ (cf., e.g., Wallrafl (1986)). Thus, there is no direct coupling between vehicle nominal motion and structural axial deformation. In this case, the equations of motion (12) and (15) in weak form simplify to

$$M \dddot{Y} + \frac{\partial^2 \dddot{Y}}{\partial S^2} - F + 2 \dot{M} \dddot{Y} = 0,$$  \hspace{1cm} (18a)

and

$$\eta^2(Y, t)[\ddot{Y} + \frac{\partial^2 \dddot{Y}}{\partial S^2}] + \int_{[0, L]} A \eta \psi'(S, t) \psi'(S, t) ds + G(u, \eta) = 0,$$  \hspace{1cm} (18b)

which are also valid for a finitely deformed beam. In (18), the equation for axial displacement and the equation for transverse displacement are coupled through the nonlinear nature of $G(u, \eta)$ for the finite deformation case. It is important to quantify the (dynamic) contact forces. In particular, these forces are crucial in studying structural response to emergency braking of a vehicle. For the basic problem considered herein, let $F_c = F_c \eta$ be the contact force acting on the wheel. Once $Y(t)$ and $u(t)$ have been solved for, the contact force can be evaluated by $F_c = \mathcal{F} - M \dot{Y}$, obtained from considering the equilibrium of forces acting on the wheel. Recall that $Y(t)$ is
evaluated using (1), (17), and with the aid of (4b). In the case of a moving magnet, the contact force $F_c$ is the required active control force that should be exerted on the magnet to maintain a constant gap.

3.3 Assumptions on Small Structural Deformation. Equations (12) and (15) form the complete set of coupled, fully nonlinear equations describing the motion of a rigid wheel moving over a flexible beam. In the present work, we consider the following additional assumptions to reduce the equations (12) and (15) to a mildly nonlinear form: (A1) $|\mathbf{u}_a, s| << 1$, for $\alpha = 1, 2$; (A2) The Bernoulli-Euler hypothesis is adopted for beam response,

$$\psi(a) = -\frac{1}{2} \int_{\partial \mathcal{A}} \left( EA \left| \mathbf{u}_s a \right|^2 + EI \left| \mathbf{u}_s s \right|^2 \right) dS,$$

where $EA$ is the axial stiffness, and $EI$ the bending stiffness; (A3) All nonlinear terms in the displacement $u^a$ are neglected in the equations for structural motion; (A4) The wheel rolls without slipping and with little influence from structural deformation,

$$\theta(Y^t, t) = \frac{Y^t}{R}, \quad \frac{\partial \theta(Y^t, t)}{\partial S} = \frac{1}{R}, \quad \theta(Y^t, t) = \frac{Y^t}{R}, \quad \theta_s(Y^t, t) = \frac{Y^t}{R},$$

$$\frac{\partial \theta}{\partial \mathbf{u}_a} = 0, \quad \text{and} \quad \frac{\partial \theta}{\partial \mathbf{u}_s a} = 0.$$  \hspace{1cm} (19b)

Note that the aforementioned assumptions are not only physically relevant in real operational conditions of the system, but carry important implications on the numerical treatment as well (see Vu-Quoc and Olsson (1988a)).

3.4 The Mildly Nonlinear Equations of Motion. Consider the structural equations of motion (15b), assumption (A3) implies that we neglect nonlinear terms in $u^a$ and $u^2$ in the fully-expanded expressions of $\ddot{g}^1$ and of $\ddot{g}^2$ obtained from using (20) and (4b) in (17). Thus, together with assumption (A1), we arrive at the approximations

$$\ddot{g}^1 = \dot{g}^1 - \ddot{R} \mathbf{u}_s, \quad \ddot{g}^2 = \dot{g}^2.$$  \hspace{1cm} (20)

Note that approximations (20) together with relations (4) when applied to $\ddot{g}^1$ and $\ddot{g}^2$ imply

$$\frac{\partial \ddot{g}^1}{\partial S} = \ddot{R}_u \mathbf{u}_s + \ddot{R}_u \mathbf{u}_s, \quad \frac{\partial \ddot{g}^2}{\partial S} = \ddot{R}_u \mathbf{u}_s + \ddot{R}_u \mathbf{u}_s,$$

for $(i, j) = (1,0), (2,0), (1,1), (0,2).$ Further, assumptions (A1) and (A3) lead to the following approximations

$$\frac{\partial \ddot{g}^1}{\partial u_{1,s}} = -\ddot{R}_u u_{1,s}, \quad \frac{\partial \ddot{g}^1}{\partial u_{2,s}} = -\ddot{R}_u u_{2,s},$$

$$\frac{\partial \ddot{g}^2}{\partial u_{1,s}} = 0, \quad \frac{\partial \ddot{g}^2}{\partial u_{2,s}} = -\ddot{R}_u u_{2,s},$$

where (22a) are obtained with the additional aid of (21) (or (2a)). As a result of (4b), (20), (22), together with assumption (A4) (i.e., (19b)), the equation for nominal motion (12) can be approximated by

$$c_1(Y^t, t) \dddot{Y}^t + c_2(Y^{t, t}) \dddot{Y}^t + c_3(Y^{t, t}) \dddot{Y}^t + c_0(Y^{t, t}) = 0,$$

where

$$c_1(Y^{t, t}) = 2M \left[ 1 - \ddot{R} u_{2,s} \right] [u_{1,s} \mathbf{u}_s(Y^{t, t})],$$

$$c_2(Y^{t, t}) = M \left[ 1 + \ddot{R} u_{2,s} + u_{2,s} \right] [u_{1,s} \mathbf{u}_s(Y^{t, t})],$$

$$c_3(Y^{t, t}) = M \left[ 1 - \ddot{R} u_{2,s} \right] [u_{1,s} \mathbf{u}_s(Y^{t, t})].$$  \hspace{1cm} (23a)

Remark 3.3. Consistency in the Formulation. The nonlinear term in $g^2$ in the equation for the nominal motion (12) is, according to (20) and (22), approximated by

$$\frac{\partial \ddot{g}^2(Y^t, t)}{\partial S} \ddot{g}^2(Y^t, t) = u_{2,s} \mathbf{u}_s(Y^{t}, t) + \ddot{R}_u u_{2,s} \mathbf{u}_s(Y^{t}, t),$$

which is also nonlinear in $u^2$. Using (4b), we obtain the term (24a) in expanded form as given in (23). This term plays an important role in representing the influence of transverse structural displacement on vehicle nominal motion at high speed. To see this, we rewrite the equation for nominal motion (18a) of Magleby model C, for $F^2 = 0$, as follows

$$M \dddot{Y}^t = u_{2,s} \mathbf{u}_s(Y^{t, t}) + \left[ F^2 - M \ddot{g}^2 \right] \ddot{g}^2(Y^{t, t}) = u_{2,s} \mathbf{u}_s(Y^{t, t}) F^2(Y^{t, t}).$$

At high speed, the amplitude of the vertical contact force $F^2$ may significantly exceed that of the vertical force $F^2$. We will present next an example with high speed vehicle motion where one has $|F^2(t)| > 1.5 |F^2|$, for some time $t$. In other words, the inertia force $M \ddot{g}^2$ could be of the same order of magnitude as that of $F^2$, and should be retained in equation (23). Hence, it is shown that the formulation would not be appropriate for high speed regime, had we systematically removed all nonlinear terms in $u^a$ from the equations of motion. This is a variance with the usual practice of complete linearization (see discussion in Kortum (1986)), which is therefore inconsistent in the present situation.

Now, applying assumptions (A1-A4), the weak form of the equations for structural motion, which is linear in the displacement $u^a$, is given by

$$\eta^1(Y^t, t) \left( -F^1 + M \dddot{Y}^t + \ddot{u}^{1}(Y^t, t) - \dddot{R} \mathbf{u}_s(Y^{t, t}) \right)$$

$$-\dddot{R} \left[ F^2 - M \ddot{g}^2 \right] \ddot{g}^2(Y^{t, t}) = u_{2,s} \mathbf{u}_s(Y^{t, t}) F^2(Y^{t, t})$$

$$+ \int_{[0, L]} A_y \eta^1(S, t) u_{1,s}(S, t) dS$$

$$+ \int_{[0, L]} E A \eta^1_s(S, t) u_{1,s}(S, t) dS = 0,$$

and

$$-\dddot{R} \eta^2_s(S, t) \left( -F^1 + M \dddot{Y}^t + \ddot{u}^{1}(Y^t, t) - \dddot{R} \mathbf{u}_s(Y^{t, t}) \right)$$

$$+ \eta^2(Y^{t, t}) \left( -F^2 + M \ddot{g}^2(Y^{t, t}) \right)$$

$$+ \int_{[0, L]} A_y \eta^2(S, t) u_{2,s}(S, t) dS$$

$$+ \int_{[0, L]} E \eta^2_s(S, t) u_{2,s}(S, t) dS = 0,$$

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for all admissible variations \( (\eta_1, \eta_2) \). Next, using the relations (4), we can recast equations (25a), (25b) to the following expanded form

\[
\begin{align*}
\left[ M_{1}(Y_1, t) \right] \left( u_{1,11}(Y_1, t) - \bar{R}u_{2,11}(Y_1, t) \right) \\
+ \int_{\Delta} A_{n}(S, t)u_{1,11}(S, t)dS + 2M_{1}(Y_1, t) \left[ u_{1,11}(Y_1, t) \right] \\
- \bar{R}u_{2,11}(Y_1, t) \left( \bar{\eta}_1(u_{1,11}(Y_1, t) - \bar{R}u_{2,11}(Y_1, t)) \right) \\
+ (\bar{\eta}_1)^2 \left[ u_{1,11}(Y_1, t) - \bar{R}u_{2,11}(Y_1, t) \right] \\
- \bar{R}(F_1 - M\bar{Y}_1) \eta_1(Y_1, t)u_{1,11}(Y_1, t) \\
+ \int_{\Delta} E\eta_1(S, t)u_{1,11}(S, t)dS = \eta_1(Y_1, t)[F_1 - M\bar{Y}_1],
\end{align*}
\]

and

\[
\begin{align*}
\left[ -\bar{R}M_{1}(Y_1, t) \right] \left( u_{1,11}(Y_1, t) - \bar{R}u_{2,11}(Y_1, t) \right) \\
+ M_{2}(Y_1, t)u_{1,11}(Y_1, t) + \int_{\Delta} A_{n}(S, t)u_{1,11}(S, t)dS \\
+ 2M\bar{Y}_1 \left[ -\bar{R}u_{2,11}(Y_1, t) \left( u_{1,11}(Y_1, t) - \bar{R}u_{2,11}(Y_1, t) \right) \right] \\
+ \eta_2(Y_1, t)u_{2,11}(Y_1, t) \left[ M\bar{Y}_1 \left[ -\bar{R}u_{2,11}(Y_1, t) \left( u_{1,11}(Y_1, t) - \bar{R}u_{2,11}(Y_1, t) \right) \right] \right] \\
- \bar{R}u_{2,11}(Y_1, t) + \eta_2(Y_1, t)u_{2,11}(Y_1, t) \\
+ M\bar{Y}_1 \left[ -\bar{R}u_{2,11}(Y_1, t) \right] \left( u_{1,11}(Y_1, t) - \bar{R}u_{2,11}(Y_1, t) \right) \\
\int_{\Delta} E\eta_2(S, t)u_{2,11}(S, t)dS = \eta_2(Y_1, t)[F_2 - M\bar{Y}_1],
\end{align*}
\]

(26a)

(26b)

for all admissible variations \( (\eta_1, \eta_2) \), where terms are grouped in square brackets according to their nature (mass, velocity, convection, and stiffness terms on the left-hand side, and applied forces on the right-hand side). Note the geometric stiffness character of the term with factor \( \bar{R}(F_1 - M\bar{Y}_1) \) and of the term with factor \( \bar{R}F_2 \) in the stiffness operators of (26a) and (26b), respectively. Even though equations (23) and (26) are the simplified versions of the fully nonlinear equations (12) and (15), according to assumptions (A1) to (A4), they remain nonlinear and coupled. Moreover, these equations in spatially discrete form are not explicit ordinary differential equations, and special algorithms must be designed for numerical computation. The system is driven by the initial conditions \( Y_1(0), \bar{Y}_1(0), u_1(S, 0), u_2(S, 0) \) and the forces \( F_1, F_2, T \) applied on the wheel.

Remark 3.3. In connection with Remark 3.3, we note that the linearized structural equations of motion (26b) contains the (low order) effect of the contact force \( F_2 = F_2 - \bar{M}\bar{u}_2 \) (the term \( \bar{M}\bar{u}_2 \) appears in (26b) in expanded form using (4b). Thus, the contact force \( F_2 \) is consistently accounted for in both equations (23) and (26).

**Remark 3.4**. With assumptions (A1-A3), equation (18b) is decoupled into an equation of motion for axial vibration and an equation of motion for the transverse vibration. But this means that the Maglev model C, unlike models A and B (see Remark 3.2), cannot be used to study effects of vehicle accelerating or braking on the axial structural response.

**4 An Illustrative Example**

In this section, an example is given to illustrate the above basic model for interaction between a vehicle, starting with different initial velocities, and a flexible supporting structure. Emphasis is focused on results which are not achievable using formulations based on the traditional assumption of known vehicle nominal motion. The results, obtained by numerical methods, correspond to the set of mildly nonlinear, coupled equations (23) and (26). We refer to Vu-Quoc and Olsson (1987, 1988a) for details and discussions on the numerical algorithms employed in solving these equations.

Consider a basic model with parameters \( M = 3000 \text{ kg}, I_w = 135 \text{ kgm}^2, R = 0.3 \text{ m}, L = 24 \text{ m}, A_s = 1250 \text{ kg/m}, E_A = 5 \times 10^9 \text{ N/m}, \text{ and } E_T = 10^9 \text{ Nm}^2 \). The beam has simple supports at its ends. The wheel is subjected to a constant vertical force \( F_1 = 600,000 \text{ N} \) (with \( F_1 = T = 0 \)), whose magnitude is about 20 times that of the weight of the wheel (acceleration of gravity \( 9.81 \text{ m/s}^2 \)), creating a maximum midspan static deflection of 0.1728 m or about \( L/140 \). The lowest flexural frequency of the beam is 2.44 Hz; its lowest axial frequency is 20.8 Hz. Initial conditions are set to: \( Y_1(0) = 0, u(S, 0) = u_n(S, 0) = 0 \) with the origin of \( S \) being coincident with the left support. The vehicle moves mainly due to its own initial velocity \( \bar{Y}_1(0) \).

**Nominal Velocity**. Figure 2 shows the variation of the nominal velocities, normalized with respect to their respective initial values (at the end of the beam) of \( Y_1(0) = 50 \text{ m/s} \) and \( 100 \text{ m/s} \), as functions of the nominal position \( Y_1 \). From this figure, one can clearly observe a loss in nominal velocity at the end of the traversing: An entry velocity of 50 m/s drops by 1.2 percent at the exit, while an entry velocity of 100 m/s drops by 0.7 percent at the exit. The peak-to-peak variations in nominal velocity for these two cases are, respectively, 1.7 percent and 1.0 percent of their initial velocities. These variations stand in contrast to traditional analyses where the velocity \( \bar{Y}_1 \) is prescribed to its initial value throughout the traversing.

The drop in velocity is related to a drop in vehicle kinetic

Fig. 2 Vehicle/structure interaction at different initial velocities:
Nominal velocity (normalized wrt initial values) versus Nominal position.
Solid line: \( \bar{Y}_1(0) = 100 \text{ m/s} \), Dotted line: \( \bar{Y}_1(0) = 50 \text{ m/s} \), Beam length \( L = 24 \text{ m} \). Transactions of the ASME
energy, as part of this initial kinetic energy is transferred to the beam; we refer to Vu-Quoc and Olsson (1987) for the details. This energy transfer, which keeps the beam in free vibration after the passage of the vehicle, effectively explains the Timoshenko paradox. We note that for a sufficiently long multiple-span structure, a vehicle moving under its initial velocity, without the aid of any other external force than a vertical one, and even in the absence of all energy-dissipative force, will experience a continuous drop in velocity as a result of this type of energy transfer (examples are given in Vu-Quoc and Olsson (1988a,b)).

It is also interesting to note that at very low speed, one has a large relative increase in velocity during the traversing. For instance, for $\dot{Y}(0) = 1$ m/s, the increase in nominal velocity is about 400 percent, i.e., the maximum velocity is about 5 m/s. As a result, the traversing time ($\approx 95$) is only about one-third of the traversing time on a rigid structure (249). This increase in velocity is, however, drastically reduced to about 10 percent for $\dot{Y}(0) = 10$ m/s (see Vu-Quoc and Olsson (1988a)).

**Structural Deflection.** The greater relative loss of velocity for $\dot{Y}(0) = 50$ m/s is due to larger vertical displacement at contact point, compared to the same displacement for $\dot{Y}(0) = 100$ m/s, as recorded in Fig. 3. Also plotted on this figure are displacement at contact point for $\dot{Y}(0) = 1$ m/s (close to a static curve) and for $\dot{Y}(0) = 10$ m/s. We note the shift of the location of maximum displacement closer to the exit as entry velocity increases.

**Contact Force.** Recorded in Fig. 4 are time histories of the vertical contact force $F_z$ for initial velocities of 50 m/s and 100 m/s. As noted in Remark 3.2, the inertial force $M\ddot{z}$ is non-negligible at high speed: For $\dot{Y}(0) = 100$ m/s, this inertial force could reach 60 percent of the vertical force $F_z$ (Fig. 4). Again, this points to the consistency of the present formulation, which is crucial for a high speed regime.

**5 Closure**

We have presented a basic building block model for analyzing the interaction between high speed vehicles and supporting flexible structures. The present formulation departs completely from traditional practice of assuming known vehicle nominal motion. Nonlinear equations of motion for the basic model, with a general form of constraints and valid for large structural deformation, are derived using Hamilton’s principle. Additional assumptions, essentially on small structural deformation, are introduced to simplify these equations to a mildly nonlinear form. The applicability of the present model is demonstrated through an example. In subsequent publications, we will present efficient algorithms to integrate the nonlinear equations of motion of the complete vehicle/structure interaction problem, and further results.

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STATE COLLEGE, PA 16804 TEL: 814-865-1655

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TITLE: PAN AMERICAN CONGRESS ON APPLIED MECHANICS (PACAM)
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BLACKSBURG, VA 24061-0219 TEL: 703-231-5641

DATE: SUMMER 1991
LOCATION: BLACKSBURG, VA
ABST: INVITED
TITLE: JUTAM SYMP ON LOCAL MECHANICS FOR COMPOSITE MATERIALS
INFO: J R REDDY, DEPT. OF ENGINEERING SCIENCE AND ENGINEERING, VIRGINIA TECH,
BLACKSBURG, VA 24061-0219 TEL: 703-231-5641

DATE: SPRING 1991
LOCATION: BLACKSBURG, VA
ABST: INVITED
TITLE: JUTAM SYMP ON LOCAL MECHANICS FOR COMPOSITE MATERIALS
INFO: J R REDDY, DEPT. OF ENGINEERING SCIENCE AND ENGINEERING, VIRGINIA TECH,
BLACKSBURG, VA 24061-0219 TEL: 703-231-5641

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