

# Stability and Hopf Bifurcation of the Maglev System with Delayed Speed Feedback Control

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**Abstract** The problem of time delay speed feedback in the control loop is considered here. Its effects on the linear stability and dynamic behavior of the maglev system are investigated. It is found that a Hopf bifurcation can take place when the time delay exceeds certain values. The stability condition of the maglev system with the time delay is acquired. The direction and stability of the Hopf bifurcation are determined by constructing a center manifold and by applying the normal form method. Finally, numerical simulations are performed to verify the analytical result.

**Key words** Maglev system, delayed speed feedback control, stability, Hopf bifurcation, center manifold, normal form

## 1 Introduction

The maglev train is a novel kind of rail vehicle that has no friction with the guideway. It is suspended beneath the guideway by the attracting electromagnetic force and is driven by linear electromotor. It has better characteristics than conventional transports, including higher speed, lower noise, easier maintenance and safety. As the maglev train is one of the important modes of transport in the future, German, Japan, China, America, and other countries have devoted much resource to develop this technology.

Gottzein and Lange<sup>[1]</sup> have systematically studied the linearized model of the maglev system based on the MBB test prototype and have presented acceleration and current feedback control algorithms. Li<sup>[2]</sup> advanced the current feedback method. He defined the “current loop” and simplified the debugging procedure. The maglev system can be easily controlled using this method. Wu *et al.*<sup>[3]</sup> incorporated elastic normal mode equation of the guideway into the system model. They investigated the effect of the elastic guideway and control parameter on the dynamic behavior of the maglev system. Zhou<sup>[4]</sup> devised the method to analyze the dynamic stability of the system by means of the Liapunov characteristic numbers and applied it to the maglev system moving over flexible guideway. The theoretic research of the maglev system greatly propels the engineering procedure<sup>[5]</sup>. A few valuable results have been obtained in the research of nonlinear dynamic and bifurcation phenomenon. For example, Shi and She<sup>[6]</sup> analyzed the Hopf bifurcation of the maglev system with PID controller, and explained the reason for the maglev system to oscillate in view of control parameter. But only few documents have been found to study its time delay systematically.

Stability, bifurcation, and chaos of the non-autonomous-retarded ordinary differential equation have been attracting researchers' attention over the past decades. Xu and Pei<sup>[7]</sup> investigated the mechanism for the action of delayed speed feedback control in non-autonomous system. The existence and stability of the Hopf bifurcation were proved. Li<sup>[8]</sup> discussed the problem of Hopf bifurcation for a class of nonlinear differential equation system with time delays. The existence and stability of bifurcation periodic solutions for this system was proved. Ji<sup>[9]</sup> proposed the stability and Hopf bifurcation condition of the two-degree-of-freedom magnetic

bearing system with time delay. Gábor Stépán<sup>[10,11]</sup> studied the bifurcation phenomenon in the traffic system, and obtained the relationship between time delay, weight function, and system bifurcation. In the above documents, nonlinear system equation was reduced to simple normal form; the dynamic character can be obtained by analyzing the normal form.

Time delay is unavoidable in every part of the maglev system. When experiments are performed, if time constant is not set properly, limit cycle will appear easily. This phenomenon is the objective of this article. We study the time delay in the control loop and discuss the effect of the time delay on the stability of the maglev system at the operating point. The Hopf bifurcation is also analyzed. This paper will provide a foundation for the future research of the dynamic behavior of the maglev system.

## 2 System modeling

The maglev system is shown in Fig. 1. The electromagnet is suspended beneath the guideway. Here  $Mg$  denotes the weight of the electromagnet, and  $F_m$  represents the electromagnetic force of the electromagnet. Define  $z_a$  and  $z_m$  as the absolute and relative displacements of the electromagnet in the vertical, respectively.  $z_h$  is the warp of the guideway to the absolute reference.  $i$  and  $v(t)$  are current and voltage of the electromagnet winding, respectively. Define guideway as the reference of the vertical displacement, the sign is positive when moving direction is descending. The electromagnet is rigid, whereas the guideway is flexible.

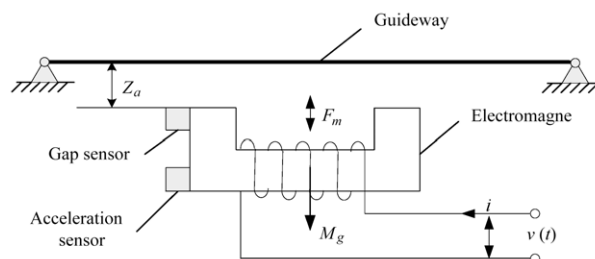


Fig. 1 Structure of the maglev system

According to [1, 2, 5], the dynamic and electromagnetic equations of the system are given as

$$v = ri + A_1 \frac{\dot{z}_m - i\dot{z}_m}{2z_m^2} \quad (1)$$

$$M\ddot{z}_a = Mg - \frac{A_1 i^2}{4z_m^2} \quad (2)$$

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$$z_a = z_m + z_h \quad (3)$$

where  $A_1 = N^2\mu_0 S_0$ ;  $\mu_0$  is magnetic permeability in vacuum,  $N$  is number of turns of coil,  $S_0$  is pole area of the electromagnet; and  $r$  is resistance of the electromagnet. The superscript “.” denotes the differential to time  $t$ .

Feedback control should be added to the port voltage of the electromagnet to keep it suspended beneath the guideway with the expected air gap  $z_e$ . According to the control algorithm given in [1],  $z_m$ ,  $\dot{z}_a$  and  $\ddot{z}_a$  are defined as the feedback state variables. Substituting (2) into (3) yields

$$\ddot{z}_a = (z_a - z_h)(g - \ddot{z}_a) \frac{4r}{A_1} - 2\sqrt{\frac{g - \ddot{z}_a}{MA_1}} v \quad (4)$$

where  $v = k_p(z_{a\tau} - z_{me} - z_{h\tau}) + k_d\dot{z}_a + k_a\ddot{z}_a$ .

The values of the state variables at the working point is  $(z_e, 0, 0)$ . Moving it to the origin, letting  $\bar{z}_a = z_a - z_e$ ,  $\dot{\bar{z}}_a = \dot{z}_a$ ,  $\ddot{\bar{z}}_a = \ddot{z}_a$ , and deleting the overbar, (4) becomes

$$\ddot{z}_a = A_2(z_a + z_e)(g - \ddot{z}_a) - A_3\sqrt{g - \ddot{z}_a}(k_p z_a + k_d \dot{z}_a + k_a \ddot{z}_a) \quad (5)$$

where  $k_p$ ,  $k_d$ , and  $k_a$  are position, velocity, and acceleration feedback control gains, respectively,  $\dot{z}_{a\tau} = \dot{z}_a(t - \tau)$  represents speed feedback control signal with time delay,  $A_2 = \frac{4r}{A_1}$ ,  $A_3 = \frac{2}{\sqrt{MA_1}}$ .

Expanding (5) up to the third order Taylor series about the operating point  $(z_a, \dot{z}_{a\tau}, \ddot{z}_a)$ , the expression is

$$\ddot{z}_a = (A_2g - \sqrt{g}A_3k_p)z_a - \sqrt{g}A_3k_d\dot{z}_{a\tau} - (\sqrt{g}A_3k_a + A_2z_e)\ddot{z}_a + f(z_a, \dot{z}_{a\tau}, \ddot{z}_a) \quad (6)$$

where  $f(z_a, \dot{z}_{a\tau}, \ddot{z}_a) = \frac{A_3k_a}{2\sqrt{g}}\ddot{z}_a + \frac{A_3k_d}{2\sqrt{g}}\ddot{z}_a\dot{z}_{a\tau} + (\frac{A_3k_p}{2\sqrt{g}} - A_2)z_a\ddot{z}_a + \frac{A_3k_a}{8\sqrt{g^3}}\ddot{z}_a^3 + \frac{A_3k_d}{8\sqrt{g^3}}\ddot{z}_a^2\dot{z}_{a\tau} + \frac{A_3k_p}{8\sqrt{g^3}}\ddot{z}_a^2z_a + O(z_a^4, \dot{z}_{a\tau}^4, \ddot{z}_a^4)$ . Because the structure of (5) is complicated, we study (6) instead of (5) in the following sections. (6) reserves nonlinearities of the system at the equilibrium and is easy to be analyzed.

### 3 Stability analysis

To analyze the stability of the system at the equilibrium, a general method is to linearize (6) at the origin and ignore the track disturbance, and then study the trivial solution of the linearized equation. Neglecting second and third order terms in (6), the linearized system equation is

$$\ddot{z}_a + (\sqrt{g}A_3k_a + A_2z_e)\ddot{z}_a + \sqrt{g}A_3k_d\dot{z}_{a\tau} + (\sqrt{g}A_3k_p - A_2g)z_a = 0 \quad (7)$$

The characteristic equation of (7) is

$$D(\lambda, \tau) = \lambda^3 + a_1\lambda^2 + b_1 + c_1\lambda \exp(-\lambda\tau) \quad (8)$$

where  $\lambda$  is the eigenvalue,  $a_1 = A_2z_e + k_a\sqrt{g}A_3$ ,  $b_1 = \sqrt{g}A_3k_p - A_2g$ ,  $c_1 = k_d\sqrt{g}A_3$ .

If the time delay in position, speed, and acceleration (PSA) feedback is not taken into account, namely set  $\tau = 0$ , then (8) becomes

$$D(\lambda, \tau) = \lambda^3 + a_1\lambda^2 + c_1\lambda + b_1 \quad (9)$$

**Lemma 1.** Roots of (9) have negative real part if and only if  $a_1 > 0$ ,  $a_1c_1 > b_1$ ,  $b_1 > 0$ .

Lemma 1 can be proved by Routh-Hurwitz stability criterion.

If control delay is considered, Lemma 1 will not guarantee the local stability of the trivial solution. If the time delay increases to a critical value, the stability of trivial solution will be changed by crossing a Hopf bifurcation.

Equation (8) is transcendental<sup>[9]</sup>, and may have an indefinite number of roots. It is difficult to get all roots explicitly. On the basis of the standard results on the stability of functional differential equations<sup>[12]</sup>, the trivial solution of (7) is stable if and only if all roots of the characteristic equation have negative real parts. Suppose the eigenvalue of (9) is  $\lambda = \alpha + i\beta$ ,  $\beta > 0$ . Substituting it into (9) yields

$$\begin{cases} \alpha^3 - 3\alpha\beta^2 + a_1\alpha^2 - a_1\beta^2 + b_1 + c_1\alpha e^{-\alpha\tau} \cos(\beta\tau) + \\ c_1\beta e^{-\alpha\tau} \sin(\beta\tau) = 0 \\ 3\alpha^2\beta - \beta^3 + 2a_1\alpha\beta - c_1\alpha e^{-\alpha\tau} \sin(\beta\tau) + \\ c_1\beta e^{-\alpha\tau} \cos(\beta\tau) = 0 \end{cases} \quad (10)$$

The fixed point will change stability when  $\text{Re}(\lambda) = 0$ , where  $\text{Re}(\lambda)$  is the real part of  $\lambda$ . This can occur in two different ways. First, a real eigenvalue passes through zero when  $b_1 = 0$ , where a simple steady bifurcation is generated. Since the trivial equilibrium is always a fixed point for (6) and the nonlinear terms are quadratic and cubic, it could be expected that pitchfork bifurcations would occur<sup>[12]</sup>.

The second situation can occur if a pair of complex eigenvalues crosses the imaginary axis, *i.e.*  $\lambda = \pm i\beta$ , a necessary condition for Hopf bifurcation. In this case, (10) is simplified to

$$\begin{cases} -a_1\beta^2 + b_1 + c_1\beta \sin(\beta\tau) = 0 \\ -\beta^3 + c_1\beta \cos(\beta\tau) = 0 \end{cases} \quad (11)$$

Equation (11) is the stable boundary of the system at the trivial fixed point. Eliminating  $\sin(\beta\tau)$  and  $\cos(\beta\tau)$  from it yields

$$\beta^6 + a_1^2\beta^4 - (2a_1b_1 + c_1^2)\beta^2 + b_1^2 = 0 \quad (12)$$

Letting  $\gamma = \beta^2$ , (12) can be written as  $g(\gamma) = \gamma^3 + a_1^2\gamma^2 - (2a_1b_1 + c_1^2)\gamma + b_1^2$ . Differentiating it with respect to  $\gamma$  yields  $\frac{dg(\gamma)}{d\gamma} = 3\gamma^2 + 2a_1^2\gamma - 2a_1b_1 - c_1^2$ . Defining  $\gamma_1$  and  $\gamma_2$  as its two roots and supposing  $\gamma_1 > \gamma_2$  and  $2a_1b_1 + c_1^2 > 0$ , we can obtain  $\gamma_1 = \frac{-a_1^2 + \sqrt{a_1^4 + 6a_1b_1 + 3c_1^2}}{3} > 0$ . If  $\gamma > \gamma_1$ , then

$\frac{dg(\gamma)}{d\gamma} > 0$ , and  $g(\gamma)$  increases with  $\gamma$  when  $\gamma \in [\gamma_1, +\infty)$ . If  $g(\gamma_1) \leq 0$  holds true, then  $g(\gamma) = 0$  holds true with  $\gamma \geq \gamma_1$ . We have the following lemma.

**Lemma 2.** If  $2a_1b_1 + c_1^2 > 0$ ,  $\gamma_1 > 0$ , and  $g(\gamma_1) \leq 0$ , then positive roots exist for (12).

From Lemmas 1 and 2, the boundary of the control parameter can be induced as

$$k_p > \sqrt{g} \frac{A_2}{A_3}, k_a > -\frac{A_2z_e}{\sqrt{g}A_3}, k_d > \frac{k_a\sqrt{g}A_3 - gA_2}{(A_2z_e + k_a\sqrt{g}A_3)\sqrt{g}A_3} \quad (13)$$

If Lemma 2 is satisfied, (10) will always have a pair of pure imaginary eigenvalues  $\pm i\beta_0$ . Solving (11), we obtain

$$\tau_n = \frac{\vartheta}{\beta_0} + \frac{2n\pi}{\beta_0}, \quad n = 0, 1, 2, \dots \quad (14)$$

where  $0 \leq \vartheta \leq 2\pi$ ,  $\cos(\vartheta) = \frac{\beta_0^2}{c_1}$ ,  $\sin(\vartheta) = \frac{a_1\beta_0^2 - b_1}{c_1\beta_0}$ .

If Hopf bifurcation exists, crossing condition must be checked<sup>[14,15]</sup>. That is, when critical eigenvalue crosses the imaginary axis, its speed is not zero. Differentiating (9) with respect to  $\lambda$  and  $\tau$  yields

$$\frac{d\lambda}{d\tau} = \frac{c_1\lambda^2 e^{-\lambda\tau}}{3\lambda^2 + 2a_1\lambda + c_1 e^{-\lambda\tau} - c_1\lambda\tau e^{-\lambda\tau}} \quad (15)$$

Setting  $\tau = \tau_n$  and  $\lambda = i\beta_0$ , the real part of (15) is

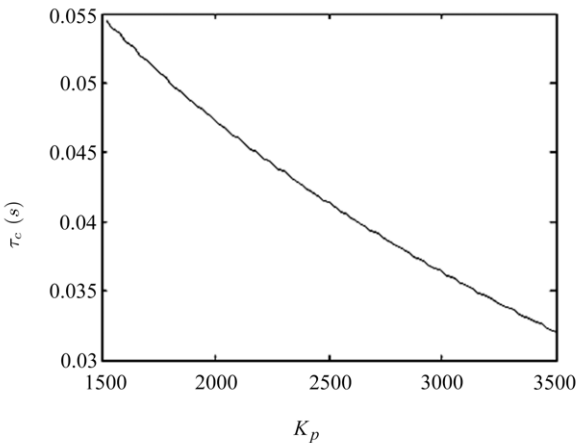
$$\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_n}^{\lambda=i\beta_0} = \frac{-c_1\beta_0^2 \cos(\beta_0\tau_n)p + c_1\beta_0^2 \sin(\beta_0\tau_n)q}{p^2 + q^2} \quad (16)$$

where  $p = -3\beta_0^2 + c_1 \cos(\beta_0\tau_n) - c_1\beta_0\tau_n \sin(\beta_0\tau_n)$ ,  $q = 2a_1\beta_0 - c_1 \sin(\beta_0\tau_n) - c_1\beta_0\tau_n \cos(\beta_0\tau_n)$ .

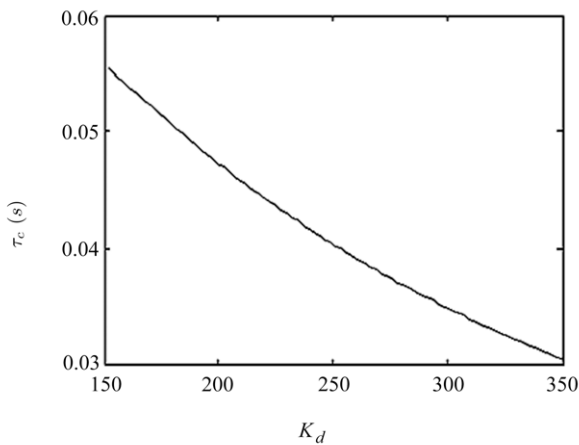
It is easy to testify that  $\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_n}^{\lambda=i\beta_0} \neq 0$ , and the crossing condition is satisfied. Similar to the conclusion given in [13], the local stability of trivial solution can be described as follows.

**Theorem 1.** If condition (13) holds true, then

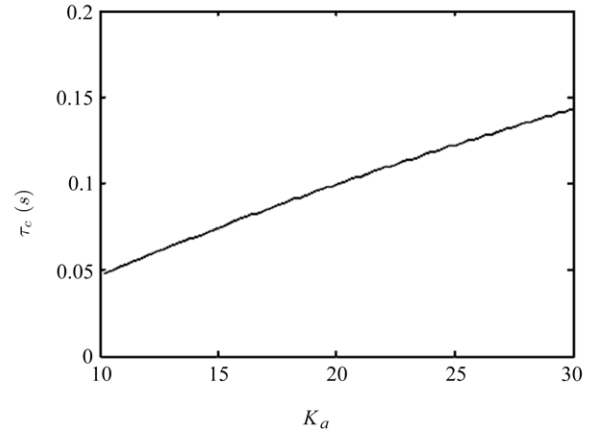
- 1) The trivial solution of (6) is asymptotically stable if  $\tau \in [0, \tau_0)$ ;
- 2) The trivial solution of (6) will not asymptotically stable if  $\tau > \tau_0$  and  $\tau \neq \tau_n$  ( $n = 0, 1, 2, \dots$ );
- 3) If  $\tau = \tau_n$  ( $n = 0, 1, 2, \dots$ ), the characteristic equation (8) has purely imaginary roots,  $\pm i\beta$ , and a single Hopf bifurcation takes place.



(a)



(b)



(c)

Fig. 2 Relationship between control parameters and critical time delay

This theorem indicates that the linear stability of the trivial solution is time delay dependent. As the length of the time delay changes, the local stability of the trivial solution may also change. From the foregoing discussion, it is easy to see that the critical value of time delay,  $\tau_c = \tau_0$ , is a function of non-dimensional feedback gains. Fig. 2 shows the relationship between time delay and control parameters. The critical time delay  $\tau_c$  increases with the increase of  $k_p$ , and decreases with the increase of  $k_d$  and  $k_a$ . The trivial solution is asymptotically stable when the corresponding parameter is in the area below the curve.

### 4 Hopf bifurcation

Hopf bifurcation refers to the situation wherein when bifurcation parameter passes through critical values, periodic motion may appear from stationary state. It has consanguineous connection with the emergency of self-oscillation and is the way to connect local stability and unstable (periodic) motion. The above analysis has shown that if we set time delay as the bifurcation parameter, when it equals to critical value, the system may lose stability and Hopf bifurcation occurs. In the experiments, if filter parameters of the control loop are not properly selected, stable limit cycle may appear. This section investigates the effect of nonlinear terms in (6) and discusses the direction and stability of Hopf bifurcation. The reason for the occurrence of limit cycle is studied. The analysis is based on center manifold/Poincaré normal form method. Detailed description of the method can be found in [10, 11].

Defining state variable as  $\mathbf{X} = [z_a \quad \dot{z}_a \quad \ddot{z}_a]^T$ , (6) can be written as

$$\dot{\mathbf{X}} = L\mathbf{Z} + R\mathbf{X}_\tau + \mathbf{F}(t, \mathbf{X}, \mathbf{X}_\tau) \quad (17)$$

where  $L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ A_2g - \sqrt{g}A_3k_p & 0 & -(\sqrt{g}A_3k_a + A_2z_e) \end{bmatrix}$ ,  
 $R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\sqrt{g}A_3k_d & 0 \end{bmatrix}$ ,  $\mathbf{F} = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}$ ,  $\mathbf{X}_\tau = \mathbf{X}(t - \tau)$ .

### 4.1 Operator differential equation<sup>[16]</sup>

To transform (17) into a functional differential equation, choose the phase space as  $\mathbf{C} = \mathbf{C}([- \tau, 0], \mathbf{R}^3)$ . For any  $\phi \in \mathbf{C}$ , define  $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$  and  $\mathbf{X}_t(\theta) = \mathbf{X}(t + \theta)$ ,  $\theta \in [-\tau, 0]$ , thus  $\mathbf{X}_t \in \mathbf{C}$ . Equation (14) can be written in the form of functional differential equation

$$\dot{\mathbf{X}} = D\mathbf{X}_t + F\mathbf{X}_t \tag{18}$$

where  $D : \mathbf{C} \rightarrow \mathbf{C}$ ,  $F : \mathbf{C} \rightarrow \mathbf{C}$ , and is defined as

$$D\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta} & \theta \in [-\tau, 0) \\ L\phi(0) + R\phi(-\tau) & \theta = 0 \end{cases} \tag{19}$$

$$F\phi(\theta) = \begin{cases} 0 & \theta \in [-\tau, 0) \\ \begin{bmatrix} 0 \\ 0 \\ f(t, \phi(0)) \end{bmatrix} & \theta = 0 \end{cases} \tag{20}$$

$$\begin{aligned} f(t, \phi(0)) = & \frac{A_3k_a}{2\sqrt{g}}\phi_3^2(0) + \frac{A_3k_d}{2\sqrt{g}}\phi_2(0)\phi_3(0) + \\ & \frac{A_3k_p}{2\sqrt{g}}\phi_3(0)(\phi_1(-\tau) - z_{h\tau}) - \\ & 2A_2(\phi_1(0) - z_h)\phi_3(0) + \frac{A_3k_a}{8\sqrt{g^3}}\phi_3^3(0) + \\ & \frac{A_3k_d}{8\sqrt{g^3}}\phi_2(0)\phi_3^2(0) - A_2gz_h + \sqrt{g}A_3k_pz_{h\tau} + \\ & \frac{A_3k_p}{8\sqrt{g^3}}\phi_3^2(0)(\phi_1(-\tau) - z_{h\tau}) \end{aligned} \tag{21}$$

where  $\phi_i$  is the elements of  $\phi$ ,  $i = 1, 2, 3$ .

From the discussion in the previous section, we know that characteristic equation (8) has a pair of pure imaginary eigenvalues  $\Lambda = \pm i\beta$  when condition (13) is satisfied and  $\tau$  equals the critical value. Therefore,  $\mathbf{C}$  can be split into two subspaces as  $\mathbf{C} = \mathbf{P}_\Lambda \oplus \mathbf{Q}_\Lambda$ , where  $\mathbf{P}_\Lambda$  is a two-dimensional space spanned by the eigenvectors of the operator  $\mathbf{D}$  associated with the eigenvalues  $\Lambda$ , and  $\mathbf{Q}_\Lambda$  is the complementary space of  $\mathbf{P}_\Lambda$ .

For any  $\psi(\xi) \in \mathbf{C}^*([0, \tau], \mathbf{R}^3)$ , we define the adjoint operator of  $\mathbf{D}$  as

$$\mathbf{D}^*\psi(\xi) = \begin{cases} -\frac{d\psi(\xi)}{d\xi} & \xi \in (0, \tau] \\ \mathbf{L}^*\psi(0) + \mathbf{R}^*\psi(\tau) & \xi = 0 \end{cases} \tag{22}$$

and the bilinear form as

$$\langle \psi, \phi \rangle = \bar{\psi}^T(0)\phi(0) + \int_{-\tau}^0 \bar{\psi}^T(\xi + \tau)R\phi(\xi)d\xi \tag{23}$$

where  $\mathbf{C}^*$  is the dual space of  $\mathbf{C}$ ,  $\phi \in \mathbf{C}$ ,  $\psi \in \mathbf{C}^*$ . Then,  $\mathbf{D}^*(0)$  and  $\mathbf{D}(0)$  are adjoint operators.

Suppose that  $\mathbf{q}_1(\theta)$  and  $\mathbf{q}_2(\theta)$  are the eigenvectors corresponding to the eigenvalue  $\lambda_1 = i\beta$ , and satisfy

$$\begin{aligned} \mathbf{D}(\mathbf{q}_1(\theta) + \mathbf{q}_2(\theta)) &= i\beta(\mathbf{q}_1(\theta) + \mathbf{q}_2(\theta)) \Rightarrow \\ \mathbf{D}\mathbf{q}_1(\theta) &= -\beta\mathbf{q}_2(\theta), \quad \mathbf{D}\mathbf{q}_2(\theta) = \beta\mathbf{q}_1(\theta) \end{aligned} \tag{24}$$

According to the definition (16), (21) has the solution

$$\begin{cases} \mathbf{q}_1(\theta) = \mathbf{S}_1 \cos(\beta\theta) - \mathbf{S}_2 \sin(\beta\theta) \\ \mathbf{q}_2(\theta) = \mathbf{S}_2 \cos(\beta\theta) + \mathbf{S}_1 \sin(\beta\theta) \end{cases} \tag{25}$$

where  $\mathbf{S}_1 = [1 \ 0 \ -\beta^2]^T$ ,  $\mathbf{S}_2 = [0 \ \beta \ 0]^T$ .

Similarly,  $\mathbf{q}_1^*$  and  $\mathbf{q}_2^*$  are supposed to be the eigenvectors corresponding to the eigenvalue  $\Lambda_1^* = -i\beta$ , and their solutions are

$$\begin{bmatrix} \mathbf{q}_1^*(\xi) \\ \mathbf{q}_2^*(\xi) \end{bmatrix} = \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix} \cos(\omega\xi) + \begin{bmatrix} -\mathbf{N}_2 \\ \mathbf{N}_1 \end{bmatrix} \sin(\omega\xi) \tag{26}$$

where  $\mathbf{N}_1$  and  $\mathbf{N}_2$  are  $3 \times 1$  vectors. The orthonormality conditions are known to be

$$\langle \mathbf{q}_1^*, \mathbf{q}_1 \rangle = 1, \langle \mathbf{q}_1^*, \mathbf{q}_2 \rangle = 0 \tag{27}$$

$\mathbf{N}_1$  and  $\mathbf{N}_2$  can be solved according to (27)

$$\mathbf{N}_1 = [n_1 \ n_2 \ n_3]^T, \mathbf{N}_2 = [n_4 \ n_5 \ n_6]^T \tag{28}$$

where  $B = [\sqrt{g}A_3k_d\tau \sin(\beta\tau) + 2\beta(A_2z_e + \sqrt{g}A_3k_a)]^2 - \sqrt{g}A_3k_d(\tau \cos(\beta\tau) + \frac{\sin(\beta\tau)}{\beta})[2\beta^2 + \sqrt{g}A_3k_d(\tau \cos(\beta\tau) - \frac{4\beta^2 + 2\sqrt{g}A_3k_d(\tau \cos(\beta\tau) - \frac{\sin(\beta\tau)}{\beta})}{\beta})]$ ,  $n_3 = \frac{4\beta^2 + 2\sqrt{g}A_3k_d(\tau \cos(\beta\tau) - \frac{\sin(\beta\tau)}{\beta})}{B}$ ,  $n_6 = -\frac{4\beta(A_2z_e + \sqrt{g}A_3k_a) + 2\sqrt{g}A_3k_d\tau \sin(\beta\tau)}{B}$ .

$n_1, n_2, n_4, n_5$  will not be used in the latter research, so their expressions are not given here.

Now we introduce new variables:  $z_1 = \langle \mathbf{q}_1^*, \mathbf{X}_t \rangle$ ,  $z_2 = \langle \mathbf{q}_2^*, \mathbf{X}_t \rangle$ , and  $\mathbf{w} = \mathbf{X}_t - z_1\mathbf{q}_1 - z_2\mathbf{q}_2$ . According to (19)-(28) and neglecting non-significant terms, Poincaré normal form of (18) can be written as

$$\dot{z}_1 = \beta z_2 + n_3(\tilde{g}_1(z_1, z_2) + \tilde{g}_2(z_1, z_2, w_1, w_2, w_3)) \tag{29}$$

$$\dot{z}_2 = -\beta z_1 + n_6(\tilde{g}_1(z_1, z_2) + \tilde{g}_2(z_1, z_2, w_1, w_2, w_3)) \tag{30}$$

$$\begin{aligned} \dot{\mathbf{w}}(\theta) &= D\mathbf{w}(\theta) + \tilde{g}_1(z_1, z_2)(n_6\mathbf{q}_2(\theta) - n_3\mathbf{q}_1(\theta)) + \\ & \tilde{g}_2(z_1, z_2, w_1, w_2, w_3)(n_6\mathbf{q}_2(\theta) - n_3\mathbf{q}_1(\theta)) + \\ & \begin{cases} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \text{if } \theta \in [-\tau, 0) \\ \begin{bmatrix} 0 \\ 0 \\ \tilde{g}_1(z_1, z_2) \end{bmatrix} & \text{if } \theta = 0 \end{cases} \end{aligned} \tag{31}$$

where  $\tilde{g}_1(z_1, z_2) = (2\beta^2A_2 + \frac{\beta^4A_3k_a}{2\sqrt{g}} - \frac{\beta^2A_3k_p}{2\sqrt{g}})z_1^2 - \frac{\beta^3A_3k_d}{2\sqrt{g}}z_1z_2 + \frac{\beta^5A_3k_d}{8\sqrt{g^3}}z_1^2z_2 + (\frac{\beta^4A_3k_p}{8\sqrt{g^3}} - \frac{\beta^6A_3k_a}{8\sqrt{g^3}})z_1^3$ ,  $\tilde{g}_2(z_1, z_2, w_1, w_2, w_3) = (\beta^2A_2 - \frac{\beta^2A_3k_p}{2\sqrt{g}})w_1(0)z_1 + (\frac{A_3k_p}{2\sqrt{g}} - \frac{\beta^2A_3k_a}{\sqrt{g}} - A_2)w_3(0)z_1 + \frac{\beta A_3k_d}{2\sqrt{g}}w_3(0)z_2 - \frac{\beta^2A_3k_d}{2\sqrt{g}}w_2(-\tau)z_1$ ,  $w_1, w_2$ , and  $w_3$  are elements of  $\mathbf{w}$ .

### 4.2 Center manifold reduction

The plane spanned by the eigenvectors  $\mathbf{q}_1$  and  $\mathbf{q}_2$  is tangential to the center manifold at the origin. This means that the center manifold can be locally approximated as a truncated power series of  $\mathbf{w}$  depending on the second order of the co-ordinates  $z_1$  and  $z_2$ :

$$\mathbf{w}(\theta) = 0.5\mathbf{h}_{20}(\theta)z_1^2 + \mathbf{h}_{11}(\theta)z_1z_2 + 0.5\mathbf{h}_{02}(\theta)z_2^2 \tag{32}$$

The unknown vectors  $\mathbf{h}_{20}$ ,  $\mathbf{h}_{11}$ , and  $\mathbf{h}_{02}(\in R^3)$  can be acquired by calculating the differentiation of  $\mathbf{w}$  and considering (29)~(31). The boundary conditions are known to be

$$Lh_{20}(0) + Rh_{20}(-\tau) + 2\beta h_{11}(0) + 2\alpha_1 t_3 = 0 \tag{33}$$

$$Lh_{11}(0) + Rh_{11}(-\tau) - \beta(h_{20}(0) - h_{02}(0)) - \alpha_1 t_4 = 0 \tag{34}$$

$$Lh_{02}(0) + Rh_{02}(-\tau) - 2\beta h_{11}(0) = 0 \tag{35}$$

The solutions of  $h_{20}$ ,  $h_{11}$ , and  $h_{02}$  can be written as

$$\begin{bmatrix} h_{20}(\theta) \\ h_{11}(\theta) \\ h_{02}(\theta) \end{bmatrix} = \begin{bmatrix} H_0 \\ 0 \\ H_0 \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \\ -H_1 \end{bmatrix} \cos(2\beta\theta) + \begin{bmatrix} -H_2 \\ H_1 \\ H_2 \end{bmatrix} \sin(2\beta\theta) + \begin{bmatrix} -2\alpha_1 t_4 - 2\alpha_2 t_3 \\ 2\alpha_1 t_3 + \alpha_2 t_4 \\ -4\alpha_2 t_3 + 4\alpha_1 t_4 \end{bmatrix} \frac{\cos(\beta\theta)}{3\beta} + \begin{bmatrix} 2\alpha_1 t_3 - 2\alpha_2 t_4 \\ 2\alpha_2 t_3 - \alpha_1 t_4 \\ 4\alpha_1 t_3 + 2\alpha_2 t_4 \end{bmatrix} \frac{\sin(\beta\theta)}{3\beta} \tag{36}$$

where  $\alpha_1 = [-n_3 \quad -\beta n_6 \quad \beta^2 n_3]^T$ ,  $\alpha_2 = [-n_6 \quad \beta n_3 \quad \beta^2 n_6]^T$ ,  $t_3 = 2\beta^2 A_2 + \frac{\beta^4 A_3 k_a}{2\sqrt{g}} - \frac{\beta^2 A_3 k_p}{2\sqrt{g}}$ ,  $t_4 = -\frac{\beta^3 A_3 k_d}{2\sqrt{g}}$ .

Substituting (36) into previous boundary conditions yields (37).

The above derivation reduced the system to the center manifold. This is a key procedure in calculating the Hopf bifurcation.

### 4.3 Poincaré normal form

$H_0$ ,  $H_1$ , and  $H_2$  can be solved from (37). Substituting them into (36),  $h_{20}(\theta)$ ,  $h_{11}(\theta)$  and  $h_{02}(\theta)$  become known variables. We can solve (31) to obtain  $w(0)$  and  $w(-\tau)$ . Substituting them into (32) and (33), the Poincaré normal form is obtained as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} + \begin{bmatrix} \sum_{\substack{j+k=2,3 \\ j,k>0}} a_{jk}^{(1)} z_1^j z_2^k \\ \sum_{\substack{j+k=2,3 \\ j,k>0}} a_{jk}^{(2)} z_1^j z_2^k \end{bmatrix} \tag{38}$$

which describes the flow restricted onto the two-dimensional center manifold. The coefficient  $a_{jk}^{(i)}$  is a known parameter. Using the coefficients of the Poincaré normal form (38), the so-called Poincaré-Lyapunov coefficient can be determined by the Bautin formula<sup>[17]</sup>

$$\Delta = \frac{1}{8\beta}(a_{20}^{(1)} + a_{02}^{(1)})(-a_{11}^{(1)} + a_{20}^{(2)} - a_{02}^{(2)}) + \frac{1}{8\beta}(a_{20}^{(2)} + a_{02}^{(2)})(a_{20}^{(1)} - a_{02}^{(1)} + a_{11}^{(1)}) +$$

$$\frac{1}{8}(3a_{30}^{(1)} + a_{12}^{(1)} + a_{21}^{(2)} + 3a_{03}^{(2)}) \tag{39}$$

Equation (39) determines the direction and stability of the Hopf bifurcation. For example, if  $\Delta < 0$  ( $\Delta > 0$ ), then the trivial solution is “attractor (repellor)”. Hopf bifurcation at the critical value is supercritical (subcritical). That is, if time delay is larger than the critical value, asymptotically stable (unstable) limit cycle arises around the unstable equilibrium.

## 5 Numerical analysis

The Poincaré normal form of (6) was obtained and the method to determine the character of Hopf bifurcation was given in Section 4. To illustrate the above conclusions, consider a system with  $N = 320$ ,  $r = 0.5\Omega$ ,  $m = 500$  kg,  $\mu_0 = 4\pi \times 10^{-7}$ ,  $z_e = 0.008$  m,  $S = 0.0047$  m<sup>2</sup>,  $k_p = 2000$ ,  $k_d = 200$ ,  $k_a = 10$ , and time delay is considered the bifurcation parameter. It is easy to know that (6) only has trivial solution and the single Hopf bifurcation takes place at  $\beta_0 = 20.3126$ ,  $\tau_c = 0.0472784$ . Substituting system parameters into (37), (39) can be written as

$$\dot{z}_1 = \beta z_2 + a_{20}^{(1)} z_1^2 + a_{11}^{(1)} z_1 z_2 + a_{02}^{(1)} z_2^2 + a_{30}^{(1)} z_1^3 + a_{21}^{(1)} z_1^2 z_2 + a_{12}^{(1)} z_1 z_2^2 + a_{03}^{(1)} z_2^3 \tag{40}$$

$$\dot{z}_2 = -\beta z_1 + a_{20}^{(2)} z_1^2 + a_{11}^{(2)} z_1 z_2 + a_{02}^{(2)} z_2^2 + a_{30}^{(2)} z_1^3 + a_{21}^{(2)} z_1^2 z_2 + a_{12}^{(2)} z_1 z_2^2 + a_{03}^{(2)} z_2^3 \tag{41}$$

where  $a_{20}^{(1)} = 281.688$ ,  $a_{11}^{(1)} = -199.891$ ,  $a_{02}^{(1)} = 0$ ,  $a_{30}^{(1)} = -24264.7$ ,  $a_{21}^{(1)} = -3146.18$ ,  $a_{12}^{(1)} = -9733.16$ ,  $a_{03}^{(1)} = -150.353$ ,  $a_{20}^{(2)} = 638.29$ ,  $a_{11}^{(2)} = -452.944$ ,  $a_{02}^{(2)} = 0$ ,  $a_{30}^{(2)} = -54982.6$ ,  $a_{21}^{(2)} = -7129.09$ ,  $a_{12}^{(2)} = -22054.8$ ,  $a_{03}^{(2)} = -340.691$ .

From (39), it is computed that  $\Delta = -14980.3$ , which means the Hopf bifurcation is supercritical. When the trivial solution of the system is unstable, the stable limit cycle may occur.

To testify the result of the analysis, a 4th-order Runge-Kutta method is used to search the numerical solution. Using the system parameters given above, the time delay is set to be  $\tau = 0.045$  and  $\tau = 0.05$ . The amplitude of the system is restricted to  $[0.017, 0.005]$  m. The system originates from 0.017 m and is expected to suspend at 0.008 m. Response curves for time history and phase trajectory are shown in Fig. 3. When  $\tau < \tau_c$ , the trivial solution is asymptotically stable. When  $\tau > \tau_c$  and  $\tau$  is near  $\tau_c$ , stable periodic motion occurs. That is the limit cycle that is induced by Hopf bifurcation. The numerical result is consistent with the theoretical analysis. Limit cycle may be brought into occurrence by delayed speed feedback control.

$$\begin{bmatrix} L + R & L + R \cos(2\beta\tau) & 2\beta I + R \sin(2\beta\tau) \\ 0 & -2\beta I - R \sin(2\beta\tau) & L + R \cos(2\beta\tau) \\ L + R & -L - R \cos(2\beta\tau) & -2\beta I - R \sin(2\beta\tau) \end{bmatrix} \begin{bmatrix} H_0 \\ H_1 \\ H_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -10\alpha_1 t_3 - 2\alpha_2 t_4 \\ -3\alpha_1 t_4 + 2\alpha_2 t_3 \\ 4\alpha_1 t_3 + 2\alpha_2 t_4 \end{bmatrix} + \frac{1}{3\beta} \begin{bmatrix} 2L(\alpha_1 t_4 - \alpha_2 t_3) + 2R(\alpha_1 t_3 - \alpha_2 t_4) \sin(\beta\tau) + 2R(\alpha_1 t_4 + \alpha_2 t_3) \cos(\beta\tau) \\ -L(2\alpha_1 t_3 - \alpha_2 t_4) + R(2\alpha_2 t_3 - \alpha_1 t_4) \sin(\beta\tau) - R(2\alpha_1 t_3 + \alpha_2 t_4) \cos(\beta\tau) \\ 4L(\alpha_2 t_3 - \alpha_1 t_4) + 2R(2\alpha_1 t_3 + \alpha_2 t_4) \sin(\beta\tau) + 4R(\alpha_2 t_3 - \alpha_1 t_4) \cos(\beta\tau) \end{bmatrix} \tag{37}$$

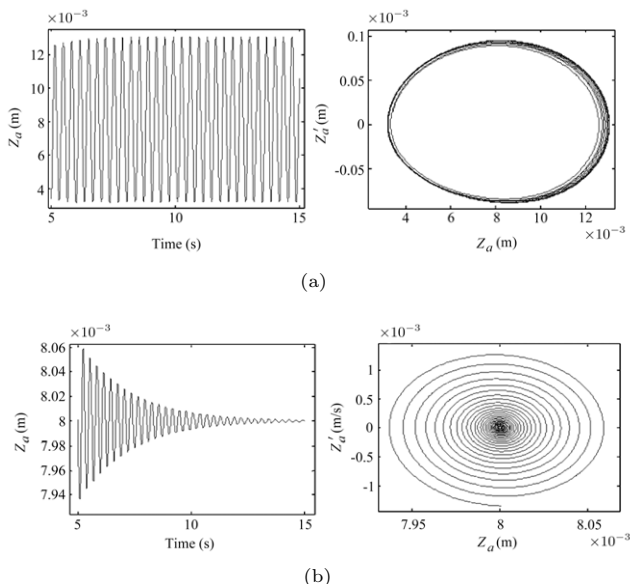


Fig. 3 Time history and phase trajectory of the system  
(a)  $\tau = 0.05$ ; (b)  $\tau = 0.045$

## 6 Conclusion

Time delay existing in the control loop has strong impact on the system performance. The relationship between control parameter and critical time delay is obtained. We also obtained the condition with which the Hopf bifurcation may be induced by time delay. To study the relationship among time delay, Hopf bifurcation and system stability, the method of center manifold/Poincaré normal form is applied. The relationship among them is shown for the first time: if time delay is much smaller than critical value, its effect on the system can be neglected; otherwise, stable or unstable periodic motion may occur, as a result of which control gains must be regulated to retain the control ability. This paper provides researchers with an in depth understanding of the maglev system. It serves as a foundation to improve and optimize the control ability of the system.

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