‘Electromaglev’ (‘active-maglev’) — magnetic levitation of a superconducting disk with a DC field generated by electromagnets: Part 3. Theoretical results on levitation height and stability

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We present Part 3 results of a comprehensive theoretical study of an ‘electromaglev’ (‘active-maglev’) system, in which a high-temperature superconducting bulk YBCO sample is levitated stably in a DC magnetic field generated by a magnet system. Field solutions have been obtained numerically to compute levitation height and define stability criteria for the superconducting disk sample. Our analysis assumes that the disk, which otherwise obeys the Bean critical-state model, traps flux when cooled in the presence of a field from the normal state to the superconducting state. Indeed it is shown that the trapped flux makes subtle and crucial changes in field distribution (and thus current density distribution) in the disk, which differ from those in a disk strictly obeying the Bean model used in the zeroth-order theory. The analysis confirms a key experimental finding that the trapped flux is another essential element for determining levitation height and ensuring tilt-free stable levitation. For stability, trapped flux is in addition to at least two degrees of freedom for spatial supercurrent flow and the profile conditions imposed on the field generated by the magnet system. Procedures to produce stable, tilt-free levitation are described. Agreement between experiment and analysis on dependence of levitation height on magnet current is quite good. The analysis also shows that to achieve stable levitation, a YBCO ring sample requires a radial build that is sufficiently thick to permit the supercurrent to flow in the radial direction. The minimum radial build required, ΔR_{rmin}, for a YBCO ring of outside radius 12.5 mm operating at 77 K is typically ~ 50 μm. An analytical expression that gives approximate values of ΔR_{rmin} has also been derived; ΔR_{rmin} depends inversely on the square of the critical current density of the superconductor. © 1998 Elsevier Science Ltd. All rights reserved

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We have been studying an ‘electromaglev’ (‘active-maglev’) system, in which a superconducting bulk sample is levitated stably in a DC magnetic field. Figure 1 shows a cross-sectional drawing of an active-maglev system used to generate both experimental and analytical results reported here. The system consists of: (1) YBCO float (disk in this figure); (2) electromagnets (inside and outside coils); and steel plates. The entire components are operated immersed in a bath of liquid nitrogen. The focus of the study has been on levitation stability, lift-to-weight ratio, and lateral stiffness. Four YBCO bulk samples have been studied: (1) disk; (2) annulus; (3) the same annulus with a permanent magnet disk placed in the hole; and (4) ring. The difference between annulus and ring is explained later. The zeroth-order theory has been developed to interpret the experimental results. Agreement between experiment and theory (zeroth order) on lift-to-weight ratio and lateral stiffness has been excellent, and it validates the basic premise of the zeroth-order theory that lift is the Lor-
entz interaction force between the supercurrent circulating in each YBCO float at the rim and the radial component of the magnetic field generated by the magnet system.

This paper presents numerical analyses that lead to field solutions for a YBCO disk exposed to a nonuniform field generated by the magnet system. The field solutions are used to compute levitation height, lift, and moment for the disk. It also presents results of an analysis on two necessary conditions required for a sample to achieve stable levitation: (1) a sufficient level of trapped flux; and (2) at least two degrees of freedom for spatial supercurrent flow. Neither has been included in the zeroth-order theory used in our previous analysis. Discussed in detail are field and critical current density distributions of a disk sample with trapped flux vis-à-vis the stability of the disk, both lateral and pitch.

In the first papers on this work, we reported unstable levitation observed in the experiment with a pancake coil wound with silver-sheathed BSCCO-2223 tape, and attributed the instability to an induced supercurrent that in the pancake coil would be restricted to flow only in the azimuthal direction as the Amperian current flowing in a permanent magnet disk. An implication is that even a YBCO ring would be unstable if its radial build is too thin; conversely, if a silver-sheathed BSCCO-2223 tape has a superconducting layer that is sufficiently thick, a one-turn coil wound with such a tape would be stable. As will be presented here we have discovered that this two-dimensionality of supercurrent distribution is a natural consequence of the magnetic flux distribution within a disk or ring that levitates stably. The analysis uses a disk as the sample configuration; some of the results are then applied to a ring.

Formulation

Equivalent current method

In the zeroth-order theory, the magnetic system consisting of the electromagnets with the steel plates placed underneath is modeled as a steel-less system, in which the steel plates are replaced with another set of electromagnets that is identical to and placed coaxially at an appropriate distance below the original set. This model generates magnetic field values that agree well in the axial component, $B_z$, but not so well in the radial component, $B_r$. In this study we have relied on a numerical method, specifically an equivalent current method (ECM) to generate accurate field values for the magnetic system. ECM is simpler to execute than more conventional methods such as the finite element method (FEM) and boundary element method (BEM). Appendix A discusses the ECM in more detail.

Bean's critical-state model

The Bean critical-state model may be used to determine the supercurrent distribution in a disk. In the zeroth-order theory, the disk is modeled as a thin slice within a long cylinder of the same radius. Accordingly, an external magnetic field, $H$, directed parallel to the axis of the cylinder penetrates into the disk from its rim, inducing a supercurrent within a thin layer called the penetration depth, which in this model is given by $H/J_c$. In reality, the field of the magnet system is nonuniform and it induces supercurrents at the three surfaces of the disk, top, bottom, and rim. Because $H$ that parallels the top and bottom surfaces is much smaller than $H$ that parallels the rim, the penetration depth of the supercurrent from the top or bottom surface is much smaller than that from the rim. Among the supercurrents induced in the disk, that looping around the rim thus contributes most to the lift. This conclusion agrees with experimental results that show virtually identical lift forces for three circular samples—disk, annulus, and ring—of the same outside diameters but with presumably different supercurrents at the top and bottom surfaces.

Finite element method (FEM)

As qualitatively described above, field profiles both inside and outside of a thin disk are quite different—particularly the $B_r$ component—from those of a thin slice within a long cylinder. The penetration depth of the disk, also an important parameter for lift, is greater than that of the thin slice. A two-dimensional finite element method (FEM) based on the current vector potential ($\mathbf{T}$) method was performed to evaluate the supercurrents of the disk at three surfaces. According to the experimental results that show critical current density, $J_c$, parallel to the $a-b$ plane is ~3 times greater than that of the $c$-axis, we may neglect the current flow across the $a-b$ planes and subdivide the disk axially into several thin wafers.

The governing equation in two dimensions, with the
assumption of zero current flow in the axial direction, is given by:

\[ n \nabla \times \frac{1}{\sigma_s} \nabla \times (T_n n) + \mu_n \frac{\partial T_n}{\partial t} + \frac{\mu_n}{4\pi} n \cdot \nabla \frac{1}{r} \overline{dS} \]  

\[ = -n \cdot \frac{\partial B_s}{\partial t} \]

where \( n \) is the directional vector normal to the surface; \( \sigma_s \) is, as discussed below, an 'appropriate' conductivity of the superconductor; \( T_n \) is the current vector potential in the \( n \)-direction; \( r \) is the distance between source and point of computation; and \( B_s \) is the ambient field generated by the magnet system.

Equation (1), derived in Appendix B, is solved numerically with FEM using the Galerkin method. For applying this conventional eddy current approach to shielding supercurrent analysis, the following constituent equations are used:

\[ J = \sigma_s E; \sigma_s = \frac{J}{|E|} (E \neq 0) \]  

\[ \frac{\partial J}{\partial t} = 0 \quad (E = 0) \]

The numerical scheme based on the Bean critical state model follows an iterative process in which the conductivity of each element is adjusted to make the amplitude of the current density in that element equal to \( J \).

In the numerical analysis, a disk, of radius \( R_d = 12.5 \text{ mm} \) and thickness \( \delta_d = 5 \text{ mm} \), is divided axially into five thin wafers, each \( 1 \text{ mm} \) thick. Each wafer is further divided radially into 49 rings and one disk at the center, each having \( 0.25 \text{ mm} \) radial build. The external field used in the analysis is nonuniform, as generated by the magnet system in the experiment. Figure 2 shows the current density distributions across the disk resting atop the magnet system at a current of 20 A (same amplitude but reverse polarities through the outside and inside coils). The solid and dashed traces correspond, respectively, to those of the first (bottom) and fifth (top) wafers and the dotted trace corresponds to that of the third (middle) wafer.

**Field-cool (FC) and zero-field-cool (ZFC) processes**

In the electromagnlev experiment, a normal-state YBCO sample may be cooled to the superconducting state by either a field-cool (FC) or zero-field-cool (ZFC) process.

**Field-cool (FC).** The FC process follows the steps described below:

1. Start energizing the electromagnets, exposing a virgin superconducting sample resting atop the magnet system to an increasing magnetic field.
2. Hold the field at \( H_{k1} \) when the virgin sample begins levitating, tilted. \( \theta_{k1} \) is the \( z \)-directed field by the magnet system at the rim of the sample.
3. Remove the sample from the magnet system and warm it up to the normal state.
4. Place the sample back atop the magnet system and let it become superconducting in the presence of \( H_{k2} \); the \( z \)-directed field by the magnet system at the rim. \( H_{k2} \) can be equal to, greater, or even slightly smaller than \( H_{k1} \). In our analysis of the disk, the \( z \)-axis induction at the rim of the top wafer at \( H_{k2} \) is designated \( B_{1o} \).
0, where $d$ is measured from the center of the coils to the bottom surface of the sample. The $z$-axis induction at the rim of each wafer at $H_{z,0}$ at which the disk just begins to levitate, is denoted $B_{z,0}$.

**Zero-field-cool (ZFC).** In the ZFC process, the sample is cooled from the normal state to the superconducting state in zero field. No trapped flux is thus induced in the sample, which remains *virgin*. Tilt-free stable levitation is not possible with the virgin sample. Note that the ZFC process is a special case of the FC process in which $H_{z,2} = 0$.

**Field and current density distributions.** Figure 4 presents examples of field and supercurrent density distributions within the disk along its $x$-axis ($-R_d \leq x \leq R_d$) at $y = 0$ for the FC (Figure 4(a) and (b)) and ZFC (Figure 4(c)) processes. Both Figure 4(a) and (b) show field distributions, $B(x)$ after steps 5(a) (the light lines, from left to right, connecting $B_{t,0}$, $B_{t,1}$, and $B_{t,2}$) and step 5(b) or 5(c) (the heavy lines, from $B_{t,0}$ at the left-hand side and $B_{t,1}$ at the right-hand side), and corresponding supercurrent distributions, $J_s$, derived from the $B$ distributions with $J_s$ assumed field-independent. In Figure 4(a), $B_{t,0}$ is less than $2B_{t,0} - B_0$ ($B_{t,0} = 2B_{t,0} - B_0$), whereas in Figure 4(b), $B_{t,0}$ is greater than $2B_{t,0} - B_0$ ($B_{t,0} > 2B_{t,0} - B_0$). The significance of the difference between the two cases is evident from both $B$ and $J_s$ distributions. When $B_{t,0} = 2B_{t,0} - B_0$ (Figure 4(a)), there is a thin annular region within the disk where $B_t$ dips below $B_{t,0}$. Within this annulus a supercurrent flows in the direction opposite to that of the supercurrent flowing at the rim; this current reversal has a great influence on stability. When $B_{t,0} > 2B_{t,0} - B_0$ (Figure 4(b)), $B_t$ and $J_s$ distributions are essentially identical to those of the ZFC process (Figure 4(c)). It may also be noted that an asymmetry in the $J_s$ distribution necessarily implies the presence of $J_s$ so as to satisfy the conservation of net current.

In this paper, the field and supercurrent density distributions within a sample are evaluated through the straight application of a numerical analysis, results of which are illustrated, for example, in Figure 4. For evaluation of a system like the present one in which spatial changes in field and supercurrent density distributions are small, FEM techniques would use an enormous amount of computation time and thus is, we believe, less suitable than the numerical analysis.

**Levitation height**

For a superconducting disk of radius $R_d$ and thickness $\delta_d$, symmetric in the $\theta$-direction with an $r$- and $z$-dependent supercurrent density, $J_d(r,z)$, and exposed to an $r$- and $z$-dependent magnetic induction, $B_d(r,z)$, lift, $F_z$, at a levitation height $d$ is given by:

$$F_z = \int_{d}^{\infty} \int_{0}^{\infty} 2\pi J_d(r,z)B_d(r,z)drdz$$

Equation (4) was solved with the disk first divided axially into five wafers, each $\delta_d/5$ thick, and then with the penetration region of the $n$th wafer ($R_d - \delta_n \leq r \leq R_d$) divided radially into 15 rings, the $n$th ring having a radial
build of $\Delta \lambda_{\mu,\nu}$. Within each of the 75 rings, $B,$ and $J_a$ are assumed constant. Thus, Equation (4) becomes:

$$F_z = \frac{2 \pi \delta_d}{5} \sum_{m=1}^{15} \sum_{n=1}^{5} \{ \Delta \lambda_{\mu,\nu} f_{\mu,\nu} f_{\nu,\mu} B_{\mu} B_{\nu} \}$$

(5)

In Bean’s critical-state model, $J_a$ in the superconductor can be $0$, $J_a$, or $-J_a$, depending on the history of field exposure, in this case, to $H_z$. Thus, $J_a$ within each of the subdivisions of the disk is: (1) $-J_a$ for $dH_z/dr > 0$; (2) 0 for $dH_z/dr = 0$; or (3) $+J_a$ for $dH_z/dr < 0$.

At a levitation height of $d$, lift $F_z$ is balanced by the weight of the disk $M_d g$, where $M_d$ is its mass, with a modification to include the buoyancy of the liquid since the disk in this study series levitates immersed in liquid nitrogen. At height, $d$, the disk is in equilibrium:

$$F_z = (M_d - V_d \rho_{LN2}) g$$

(6)

where $V_d$ is the volume of the disk and $\rho_{LN2}$ is the density of liquid nitrogen.

**Lateral stability**

Lateral stability is the ability of a levitated sample to return to its original equilibrium position when displaced by a small distance in the lateral direction. In our analytical model of the disk under consideration, the disk, already divided into 75 rings, is further dissected azimuthally into 12 equal sections, as shown in Figure 5(a). This is to take into account the asymmetry of the magnetic field and current distributions caused by a displacement of the disk by $\Delta x$ in the $x$-direction (Figure 5(b)). Then the total lateral force $F_y$ is given by:

$$F_y = \frac{\pi \delta_d}{30} \sum_{m=1}^{15} \sum_{n=1}^{5} \sum_{\mu,\nu} \{ \Delta \lambda_{\mu,\nu} f_{\mu,\nu} f_{\nu,\mu} B_{\mu} B_{\nu} \} \cos \theta_y$$

(7)

where $\theta_y = \frac{\pi}{6} \left( p - \frac{1}{2} \right)$, as indicated in Figure 5(a), is in the $x-y$ plane and measured from the $+x$ axis. For lateral stability, the following condition is required: $\partial F_y / \partial x < 0$.

**Pitch stability**

Pitch stability is the ability of a levitated sample to return to its original tilt-free orientation when it is rotated by a

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**Figure 4** Examples of field and supercurrent distributions within the disk along the $x$-axis ($-R_d \leq x \leq R_d$) at $y = 0$. (a) FC process in which $B_{z0} \leq 2B_0 - B_{z0}$; (b) FC process in which $B_{z0} > 2B_0 - B_{z0}$; (c) ZFC process.

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**Figure 5** (a) Disk placed at the $x-y-z$ coordinates and divided into 12 equal sections in the azimuthal direction. (b) Disk position for lateral stability analysis, showing a displacement of $dx$ in the $+x$ direction. (c) Disk position for pitch stability analysis, showing a rotation of $\theta_y$ about the $-y$-axis.
small angle about the y-axis—more generally about the r-axis. When rotated by a small angle $\theta_r$ about the y-axis, the moment, $M_r$, is generated by both lift $F_r$ and lateral force $F_s$ acting on the sample. For a small angle of rotation, the moment by the lateral force is much less than that by lift and it may be ignored. $M_r$ due to $F_r$ on the disk under consideration rotated by $\theta_r$ about the y-axis (Figure 5(c)) is thus given by:

$$M_r = \frac{\pi \delta \xi}{30} \sum_{m=-1}^{1} \sum_{n=-1}^{1} \sum_{l=1}^{12} \left( \Delta \lambda_{m,n,p} r_{m,n,p} \right)$$

$$\times J_{1l} r_{m,n,p} \sin \theta_r \cos \theta_r \cos \theta_c \{\cos \theta_r \cos \theta_c \}$$

where, again, $\theta_r = \frac{\pi}{6} \left( p - \frac{1}{4} \right)$. For pitch stability, the required condition is: $\partial M_r / \partial \theta_r < 0$.

**Results and discussion**

**Magnetic field**

In the FC process, $B_{0r}$ (trapped field, after step 4) as well as $B_{0c}$ (magnetic field at the rim with the external field zero, after step 5(a)) are the key parameters required to determine the penetration depth. $B_{0c}$ is estimated iteratively as follows.

(a) Set $B_{0c}$ zero initially: $B_{0c} = 0$.
(b) Determine a penetration depth with $B_{0c} = 0$ and $B_{0c}$.
(c) Compute $B_{0c}$ using the net supercurrent in the penetration depth of step (b).
(d) Recompute the penetration depth with $B_{0c}$ and $B_{0c}$ determined in step (c).
(e) Repeat steps (c) and (d) until $B_{0c}$ converges.

The above sequence was used to determine $B_{0c}$ in the presence of an external field generated by the magnet system. The trapped field $B_{0r}$ is approximated to be the same as the external induction of the magnet system at a field of $B_{0c}$.

**Levitation height**

*Figure 6* presents two sets of levitation height $d$ vs. $I$ (magnet current) data, one for the disk and the other for the ring. Circles and triangles, respectively, for the disk and ring are experimental; solid (disk) and dotted lines (ring) are analytical. (Results based on the zeroth-order theory are given by the dashed and double-dashed lines, respectively.

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**Figure 6** Two sets of levitation height $d$ vs. magnet current $I$ data, one for the disk and the other for the ring. Circles (disk) and triangles (ring) are experimental; solid (disk) and dotted lines (ring) are analytical; and dashed (disk) and double-dashed (ring) lines are based on the zeroth-order theory.

**Figure 7** $d$ vs. $I$ diagrams indicating stable regions for three combinations of $I_1$ (outside coil current) and $I_2$ (inside coil current): (a) disk; and (b) ring. In each diagram the stable region has three outer boundaries—the solid ($I_1 = -I_2$ used in the experiment), dashed ($I_1 = 0.5I_1$, and double-dashed ($I_2 = 0$). Also shown are $d$ vs. $I$ plots, experiment (circles for the disk and squares for the ring) and present analysis (dotted lines for both disk and ring).

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for the disk and ring. The computed lines—this analysis and zeroth-order theory—include the buoyancy effect of liquid nitrogen. Agreement between experiment and theory (this analysis) is reasonable. Note that both experimental and analytical curves show a clear-cut transition. The discrepancy is due chiefly to an overestimation in the analysis of the trapped flux, $B_{\phi}$; agreement should be improved with a trapped flux smaller than $B_{\phi}$.

Agreement between theory and experiment in Figure 6 is better with the zeroth-order theory than with the present analysis. This should not lead to the conclusion that the zeroth-order theory is as good as or even better than the present analysis. As pointed out earlier, the zeroth-order theory does not take into account the presence of trapped flux in a sample nor the irreversibility of supercurrent distribution. Thus, the theory can predict quite accurately the dependence of levitation height with respect to magnet current, provided it is adjusted to a particular data point, as illustrated in Figure 6; it is unable to predict correct levitation height for a sample with an arbitrary value of trapped flux. Lack of trapped flux in the theory also makes it unsuitable to deal with lateral and pitch stabilities in a more general way.

The present analysis has also been applied to determine the magnet current corresponding to $H_{\phi0}$ for each sample. As described earlier in connection with the field-cool (FC) process, $H_{\phi0}$ is the minimum $z$-axis field at the rim of the sample required to ensure tilt-free stable levitation (step 5(b)). According to this analysis, currents to generate $H_{\phi0}$ are 8.1 A and 5.7 A, respectively, for the disk and ring; respective currents determined in the experiment are 8 A and 6 A. Agreement between theory and experiment is quite good.

**Lateral and pitch stability**

Figure 7 shows $d$ vs. $I$ graphs of stable regions for the disk (Figure 7(a)) and ring (Figure 7(b)) samples, for three sets of $I_1$ and $I_2$, currents, respectively, for outside and inside coils. In each graph, the stable region is bounded by the solid, dashed, and double-dashed lines, respectively, for sets of $I_2 = I_1$ (experiment), $I_2 = -0.5I_1$, and $I_2 = 0$. In addition, each graph also includes $d$ vs. $I$ plots, experiment (circles or squares) and this analysis (dotted lines), for both samples. Agreement between experiment and analysis is fair. Results of Figure 7 imply that stable levitation is achievable without inside coil, which is energized to generate a magnetic field directed opposite from that of outside coil and extend the "magnetic well" ($\partial B/\partial r > 0$) profile over a larger axial span than otherwise possible with outside coil alone.

Figure 8 shows three sets of schematic drawings for...
\( B_1(x), J_1(x) \), and \( f_1(x) \) distributions across the disk at \( y = 0 \) with the disk at the center. \( f_1(x) \) represents force density in the \( x \)-direction; here only its polarity, not amplitude, is significant. The distributions are for FC processes (Figure 8(a and b)) and the ZFC process (Figure 8(c)). In each set, the penetration region of the disk is divided into annular zones, in Figure 8(a), for example, into six zones, labelled 1 to 6. Because the external field is symmetric about the \( z \)-axis, so are the \( B_1(x), J_1(x) \), and \( f_1(x) \) distributions in each set. However, depending on the size of \( B_o \) relative to \( 2B_o - B_0 \), i.e., \( B_o = 2B_o - B_0 \) (as in Figure 8(a)) or \( B_o > 2B_o - B_0 \) (as in Figure 8(b)), there are subtle differences in these distributions between the two FC processes, as discussed earlier in connection with Figure 4.

Figure 9 shows similar sets of drawings for the disk displaced by \( dx \) in the \( x \)-direction: Figure 9(a) for \( B_o = 2B_o - B_0 \) (FC1 process) and Figure 9(b) for \( B_o > 2B_o - B_0 \) (FC2 process); and Figure 9(c) for the ZFC process. \( B_o \) is the flux at the rim of the disk upon displacement. \( B_1(x) \) in Figure 8 is the flux at the rim of the disk in equilibrium.) Because \( B_1(x) \) is now asymmetric, so are \( J_1(x) \) and \( f_1(x) \). Note that in each set the displacement: (1) creates a new annular zone (labelled 0) in the right-hand side; (2) decreases zones 1 and 5; and (3) increases zone 6. (For the sake of brevity, henceforth a ‘zone’ refers to the size of a zone; also FC and ZFC refer, respectively, to the FC and ZFC processes.) These changes all contribute to enhance the net lateral restoring force, implying that the condition of \( \partial B_1/\partial y > 0 \) that causes these changes ensures lateral stability. Note that the ratio of zone 2 to zone 1 in \( J_1 \) distribution, because zone 2 current generates a restoring force and zone 1 current an unstabilizing force, is important for lateral stability. Because zone 2 of FC1 is much smaller than that of ZFC, the ratio of zone 2 to zone 1 is much smaller for FC1 than that for ZFC: for lateral stability ZFC thus provides a condition more favorable than FC1. As noted earlier that an asymmetry in the \( J_1 \) distribution necessarily implies the presence of \( J_1 \), however, has no direct impact on stability.

Figure 10 shows three sets of schematic drawings for distributions across the disk at \( y = 0 \) of \( B_1(x), J_1(x) \), and \( m_1(x) \) (moment density about the \( y \)-axis) with the disk in equilibrium: Figure 10(a) for \( B_o = 2B_o - B_0 \) and Figure 10(b) for \( B_o > 2B_o - B_0 \), both FC; and Figure 10(c) for ZFC. Note that the \( m_1 \) distribution indicates only polarities (the + sign for moment density in the + \( y \)-direction and the − sign for moment in the − \( y \)-direction). As before, the penetration region of the disk for each FC is divided into six annular zones, 1 to 6. As in Figure 8 the sets have symmetrical distributions along the \( x \)-axis. Also as in Fig-

![Figure 9](https://example.com/figure9.png)

**Figure 9** Three sets of drawings for \( B_1(x), J_1(x) \), and \( f_1(x) \) (force density in the \( x \)-direction) distributions across the disk at \( y = 0 \) with the disk displaced by \( dx \) in the \( x \)-direction: (a) FC process in which \( B_o = 2B_o - B_0 \); (b) FC process in which \( B_o > 2B_o - B_0 \); (c) ZFC process.
There are subtle differences in distributions between those in Figure 10(a) and those in Figure 10(b).

Figure 11 shows similar sets of drawings with the disk rotated by angle \( \theta \) about the \(-y\)-axis: Figure 11(a) for \( B_{x0} = 2B_{x0} - B_0 \) and Figure 11(b) for \( B_{x0} > 2B_{x0} - B_0 \) both FC processes; and Figure 11(c) for the ZFC process. As in Figure 9, because \( B_0(x) \) is asymmetric, so are \( J_0(x) \) and \( m_0(x) \). Note that in each distribution the displacement: (1) creates a new annular zone (labelled 7); (2) decreases zones 2 and 6; and (3) increases zone 1. These changes, together with an increase of \( B_0 \) in zone 1 and decreases of \( B_0 \) in zones 2 and 6 (Figure 11(a)) and in zone 6 (Figure 11(b)), all contribute to make the disk pitch-stable. On the other hand, a combination of an increase of \( B_0 \) in zone 2 and decreases of \( B_0 \) in zones 5 and 7 makes the disk pitch-unsatable. Trapped flux is a decisive factor in making each of these rotation-induced changes, \( B_0 \) and zone size, either stabilizing or destabilizing. Thus, when trapped flux is small, which applies also to ZFC, zone 2 becomes generally much larger than zone 1, and zone 5 is much larger than zones 6 and 7, making \( B_0 \) effects dominate zone effects and hence makes the disk pitch unstable. (Note that the distributions shown in Figures 4, 8–11 are all schematic and do not present relative sizes accurately; some are drawn much larger than the actual sizes for the sake of clarity.) Conversely, the greater the trapped flux, the greater the zone effects, making the disk pitch stable, provided that the trapped flux is sufficiently great. As stated earlier, a key parameter is \( H_{x0} \), which is the trap-flux-inducing field in step 4 of the FC process. Note that \( H_{x0} \) must be at least \( H_{x0} \).

Minimum radial build

Among the four samples examined, the ring has been judged best based on the lift-to-weight criterion. Here we may distinguish a ring from an annulus in terms of its radial build, \( \Delta R = R_2 - R_1 \), where \( R_1 \) and \( R_2 \) are the inside radius of the ring; we may call an annulus a ring if \( \Delta R < R_2/2 \) and an annulus an annulus if \( \Delta R > R_2/2 \). If a ring is the best, the next obvious question is: how small can its radial build be? As demonstrated in the experiment, a BSCCO-2223 tape solenoid is unstable, perhaps in part because, unlike the YBCO annulus and ring, it is multi-turned and most likely because the effective thickness of the superconducting layer of the tape is too thin to satisfy another requirement for stable levitation: at least two degrees of freedom for spatial supercurrent flow. We shall thus now examine the radial build of the ring, specifically its minimum thickness to satisfy this spatial flow requirement.

Figure 12 illustrates two sets of \( B_0(x) \) and \( J_{0}(x) \) distributions for ring 1 (Figure 12(a)) and ring 2 (Figure 12(b)) having the same outer radius \( (R_2) \), thickness \( (d) \), and the
same level of trapped flux but with different radial thicknesses, ring 1 being thicker (and thus heavier) than ring 2. To achieve a given levitation height, ring 1 thus requires an external field greater than does ring 2, making, as seen in Figure 12(a and b), $B_n$ for ring 1 greater than that for ring 2. This in turn results in the $J_n$ zone of ring 1 being greater than that of ring 2, making ring 1 pitch-unstable and ring 2 pitch-stable. The condition that determines pitch-stability (or instability) is the relative magnitude of $B_n$ in comparison with that of $B_{elim}$ which is defined in the figure. Thus, stability is ensured if $B_n < B_{elim}$, and vice versa. This condition has an important implication: a ring, because it is lighter, is generally more pitch-stable than a disk of the same outer radius and thickness. Because the ratio of $B_1/B_{elim}$ depends not only on trapped field but also on the weight of the ring, the required minimum thickness for radial build to ensure pitch stability, $\Delta R_{min}$, may be estimated from lift, $F_z$, and the weight of the ring, $M$, $g$, where $M$ is the mass of the ring.

As an illustration, Figure 13 shows $B_z(x)$ distributions for three rings, rings 1 (Figure 13(a)), ring 2 (Figure 13(b)), and ring 3 (Figure 13(c)), all having the same thickness but different $\Delta R$. Here, it is assumed that each ring has just the right amount of trapped flux to satisfy the condition $F_z = M g$ and thus is levitated stably at a given height. Ring 1, having the largest $\Delta R$ among the three, is thus the heaviest, and models the ring used in our experiment. It requires a large $B_{elim}$ to begin levitation. Ring 2, having a smaller $\Delta R$ than ring 1, models a lighter ring. Ring 2 has the same amount of trapped field ($B_n$) as ring 1, thus requiring a smaller $B_n$ than ring 1, which, in turn, gives rise to, as evident from Figure 13(a and b), a $B_z(x)$ distribution that is subtly different from that of ring 1. The inside wall area of ring 2 now carries a supercurrent that flows in the direction opposite to the main supercurrent, resulting in a net reduction in lift. We may eliminate this negative supercurrent by reducing the trapped flux. Ring 3 models such a ring: its radial build is equal to a minimum level required for stable levitation, $\Delta R_{min}$. For ring 3 with an outer diameter of 25 mm and thickness of 5 mm, $\Delta R_{min}$ becomes $\sim 50 \mu m$, comparable with the thickness of BSCCO-2223 layers in mono-tape silver-sheathed tapes. The BSCCO-2223 solenoid that proved unstable in our study was wound with multi-tape BSCCO-2223 tape having the BSCCO-2223 layers each estimated to be one order of magnitude smaller than 50 $\mu$m. To achieve stable levitation with a one-turn ring wound of such a tape, the superconductor layer of the tape must be $\sim 100 \mu m$ and thicker.

Bean’s critical-state model may also be used to estimate $\Delta R_{min}$. Here, we may assume both a supercurrent density

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Figure 11 Three sets of schematic drawings for $B_z(x)$, $J_n(x)$, and $m_z(x)$ (moment density about the $y$-axis) distributions across the disk at $y = 0$ with the disk rotated by angle $\theta_r$ about the $-y$-axis; (a) FC process in which $B_{z0} < 2B_n - B_n$; (b) FC process in which $B_{z0} > 2B_n - B_n$; (c) ZFC process.
Figure 12. Two sets of $B_z(x)$ and $J_c(x)$ distributions for two rings of the same outer diameter but with different radial thicknesses. (a) Ring 1; (b) Ring 2. Ring 1 is thicker (and thus heavier) than ring 2.

of $J_c$ and the radial component of initial magnetic flux density within the superconductor, $B_{0r}$ to be constant. An expression for lift, $F_z$, for a ring with outside radius of $R_0$ and inside radius of $R_i$, may be given by:

\[ F_z = \pi (R_0^2 - R_i^2) \delta \mu B_{0r} J_c \]  

(9)

Because our levitation experiments\(^1\)–\(^3\) have all been performed with each YBCO sample immersed fully in a bath of liquid nitrogen boiling at 77 K, the buoyancy effect of the liquid nitrogen must be included. The effective weight of the ring, $\{M, g\}_{\text{eff}}$, is thus given by:

\[ \{M, g\}_{\text{eff}} = \pi (R_0^2 - R_i^2) \delta \mu (\rho_d - \rho_{\text{LN2}}) g \]  

(10)

where $\rho_d$ and $\rho_{\text{LN2}}$ are densities, respectively, of the material of the ring (6400 kg/m\(^3\) for YBCO) and liquid nitrogen (807 kg/m\(^3\)). From the equilibrium condition, $F_z = \{M, g\}_{\text{eff}}$, we obtain an expression for $B_{0r}$:

\[ B_{0r} = \frac{(\rho_d - \rho_{\text{LN2}}) g}{\mu J_c} \]  

(11)

Both magnetic flux densities, $B_{0r}$ and $B_{0a}$, are proportional to the magnet current. Therefore we may express $B_{0r}$ in terms of $B_{0a}$:

\[ B_{0r} = CB_{0a} \]  

(12)

where $C$ is a proportionality constant. From the $B_z(x)$ diagram of ring 3, shown in Figure 13(c), trapped field $B_{0r}$ is clearly one half of $B_{0a}$, i.e., $B_{0r} = B_{0a}/2$. The penetration depth of the supercurrent for this ring (Figure 13(c)), $\lambda_{kr}$, is equal to $R_d - R_{in}$, which is, by definition, equal to $\Delta R_{\text{min}}$ here.

\[ \lambda_{kr} = R_d - R_{in} = \Delta R_{\text{min}} \]  

(13)

Using Bean's critical-state model, we have:

\[ \Delta R_{\text{min}} = \frac{B_{0a}}{\mu_0 J_c} \]  

(14)

Combining Equations (11)–(14), we have an expression for $\Delta R_{\text{min}}$:

\[ \Delta R_{\text{min}} = \frac{(\rho_d - \rho_{\text{LN2}}) g}{2 \mu_0 J_c^2} \]  

(15)

In our system, $C$ is 0.099 and inserting appropriate values for parameters of Equation (15), e.g., $J_c = 10^9$ A/m\(^2\), we obtain: $\Delta R_{\text{min}} = 20 \mu$m, which is of the same order as ~50 $\mu$m computed in the numerical analysis. This discrepancy of ~30 $\mu$m comes about because here $\Delta R_{\text{min}}$ is derived without meeting the condition that trapped flux must be of a sufficient level for pitch stability. Note that $\Delta R_{\text{min}}$ depends inversely on $J_c^2$.

Conclusions

Field solutions have been obtained numerically to compute levitation height and define stability criteria for a superconducting disk sample in an 'electromaglev' ('active-maglev') system. Our analysis assumes that the disk, which otherwise obeys the Bean critical-state model, traps flux when cooled in the presence of a field from the normal state to the superconducting state. The analysis confirms a key experimental finding that the trapped flux is another essential element for determining levitation height and ensuring tilt-free stable levitation. For stability, trapped flux is in addition to at least two degrees of freedom for spatial supercurrent flow and the profile conditions imposed on the field generated by the magnet system.
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An analytical expression that gives approximate values of $\Delta R_{\text{min}}$ has also been derived: $\Delta R_{\text{min}}$ depends inversely on the square of the critical current density of the superconductor.

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References


Appendix A

Equivalent current method

The magnetization vector $\mathbf{M}$ is derived as follows. To obtain a formula for the magnetic flux density in the presence of steel plates, we let $\mathbf{m}_i$ be the magnetic dipole moment of an atom. If there are $n$ atoms per unit volume, we define a magnetization vector, $\mathbf{M}$ [A/m], as:

$$M = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{n} m_i}{\Delta V} \quad \text{(A1)}$$

Figure 13 $B_z(r)$ distributions for three rings having the same thickness but different values of $\Delta R$: (a) ring 1; (b) ring 2; and (c) ring 3

experiment and analysis on the dependence of levitation height on magnet current is quite good. The analysis also shows that to achieve stable levitation, a YBCO ring sample requires a radial build that is sufficiently thick to permit the supercurrent to flow in the radial direction. The minimum radial build required, $\Delta R_{\text{min}}$, for a YBCO ring of outside radius 12.5 mm operating at 77 K is typically $\sim 50$ $\mu$m.
which is the volume density of magnetic dipole moment. The magnetic dipole moment \(d\mathbf{m}\) of an elemental volume \(dV\) is \(d\mathbf{m} = \mathbf{M} dV\) that produces a vector magnetic potential:

\[
d\mathbf{A} = \frac{\mu_o}{4\pi} \frac{\mathbf{M} \times \mathbf{r}}{r^3} dV
\]

(A2)

Thus:

\[
\mathbf{A} = \int_V \mathbf{A} = \frac{\mu_o}{4\pi} \int_V \mathbf{M} \times \mathbf{r} \mathbf{r}^{-3} dV
\]

(A3)

where \(V\) is the volume of the magnetized steel plate. We now use the vector identity to write:

\[
\mathbf{M} \mathbf{r}^{-1} = \frac{1}{r} \mathbf{r} \mathbf{r} \mathbf{r}^{-1} \mathbf{M} - \mathbf{r} \mathbf{r} \mathbf{r}^{-1} \mathbf{M} = \mathbf{M} - \mathbf{r} \mathbf{r} \mathbf{r}^{-1} \mathbf{M} \mathbf{r}^{-1}
\]

(A4)

and expand the right-hand side of Equation (3) into two terms:

\[
\mathbf{A} = \int_V \frac{\mu_o}{4\pi} \mathbf{M} \mathbf{r}^{-1} \mathbf{r} \mathbf{r}^{-3} dV + \frac{\mu_o}{4\pi} \int_S \mathbf{M} \mathbf{a}_n dS
\]

(A5)

Using a vector identity, we can change the volume integral of the curl of a vector into a surface integral. We thus have, from Equation (5):

\[
\mathbf{A} = \frac{\mu_o}{4\pi} \int_V \mathbf{M} \mathbf{r}^{-1} \mathbf{r} \mathbf{r}^{-3} dV + \frac{\mu_o}{4\pi} \int_S \mathbf{M} \mathbf{a}_n dS
\]

(A6)

where \(\mathbf{a}_n\) is the unit outward normal vector from \(dS\) and \(S\) is the surface enclosing the volume \(V\). The vector magnetic potential is also expressed in terms of current density \(\mathbf{J}\):

\[
\mathbf{A} = \frac{\mu_o}{4\pi} \int_V \mathbf{J} \mathbf{r}^{-3} dV
\]

(A7)

A comparison of the expressions on the right-hand side of Equation (6) with the forms of \(A\) in Equation (7) suggests that the effect of the magnetization vector is equivalent to both current density \(\mathbf{J}\) and surface current density \(\mathbf{K}\):

\[
\mathbf{J} = \nabla \times \mathbf{M}
\]

(A8)

\[
\mathbf{K} = \mathbf{M} \times \mathbf{a}_n
\]

(A9)

Using the magnetic flux density \(\mathbf{B}\) and the magnetization vector \(\mathbf{M}\), we obtain:

\[
\mathbf{M} = \frac{1}{\mu_0} \mathbf{B} - \frac{1}{\mu} \mathbf{B}
\]

(A10)

where \(\mu\) is the absolute permeability. The vector magnetic potential given by the magnetization vector can be written as:

\[
\mathbf{A} = \frac{\mu_o}{4\pi} \int_V \frac{\mathbf{M} \times \mathbf{r}}{r^3} dV
\]

(A11)

After substituting Equation (11) into \(\mathbf{B} = \nabla \times \mathbf{A}\), we obtain an expression for \(\mathbf{B}\):

\[
\mathbf{B} = \mathbf{B} + \frac{\mu_o}{4\pi} \nabla \times \left( \frac{\mathbf{M} \times \mathbf{r}}{r^3} \right) dV
\]

(A12)

where \(\mathbf{B}\) is the magnetic flux density generated by the electromagnets placed underneath the sample. Combining Equation (10) and Equation (12), we have a final governing equation:

\[
\mathbf{M} - \frac{1}{\mu_0} \frac{1}{\mu} \mathbf{B} = \frac{1}{\mu_0} \frac{1}{\mu} \mathbf{B}
\]

(A13)

The following approximations were used in solving Equation (13): (1) the steel plates are divided into infinitesimal annular ring elements; and (2) the magnetization vector \(\mathbf{M}\) is constant within each element. The approximations make the term that includes the magnetization volume current density zero. These approximations introduce little error to Equation (13). The magnetic flux density \(\mathbf{B}\) is calculated numerically by using Biot–Savart's law over the volume of conductor in the electromagnets.

**Appendix B**

**Current vector potential method (FEM)**

The governing equation in two dimensions assumes the current distribution in the axial direction to be zero and is derived as follows.

The current vector potential \(\mathbf{T}\) is defined by:

\[
\nabla \times \mathbf{T} = \mathbf{J}
\]

(B1)

\[
\nabla \cdot \mathbf{T} = 0 \text{(Coulomb gage)}
\]

(B2)

\[
\mathbf{n} \times \nabla = 0 \text{ (on the surface)}
\]

(B3)

where \(\mathbf{n}\) is the unit vector on the surface. Using Helmholtz’s formula, the governing equation is derived as follows:

\[
\nabla \times \frac{1}{\sigma} \nabla \times \mathbf{T} + \frac{\mu_o}{4\pi} \frac{\partial \mathbf{T}}{\partial t} + \frac{\nabla}{\sigma} \cdot \nabla \frac{1}{\sigma} \mathbf{T} \cdot \nabla dS = - \frac{\partial \mathbf{B}_0}{\partial t}
\]

(B4)

where \(T, r,\) and \(B_0\) are the normal component of \(\mathbf{T}\), the distance between surface and field point, and the applied flux density by the magnet system, respectively. The governing equation in two dimensions, with an assumption that the current distribution in the axial direction is zero, may be rewritten from Equation (4) as:

\[
\mathbf{n} \nabla \times \frac{1}{\sigma} \nabla \times (T \mathbf{n}) + \frac{\mu_o}{4\pi} \frac{\partial T}{\partial t}
\]

(B5)
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\[ + \frac{\mu_0}{4\pi} \int_S \frac{\partial T_\alpha}{\partial t} \nabla (1) r dS = - \mathbf{n} \cdot \frac{\partial \mathbf{B}_0}{\partial t} \]

Note that Equation B(5) is identical to Equation (1) in the main text.